Lebesgue measure of Julia, Fatou and non-escaping sets

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Abstract

Iteration theory of meromorphic functions, which originates in the study of rational iteration by Fatou and Julia around 1920, is today an active research topic. The key concept in studying the behaviour of a meromorphic function f under iteration is the partition of the complex plane in two parts: The Fatou set $\mathcal{F}(f)$ where all iterates f^n are defined and form a normal family in the sense of Montel; and its complement, the Julia set $\mathcal{J}(f)$. In this thesis we are concerned with Lebesgue measure of Fatou and Julia sets, considering two contrasting phenomena. First we study meromorphic functions whose Julia set has measure zero, then we show that the Fatou set of certain entire functions has finite measure.

The postsingular set of a meromorphic function f is defined as the closure of the set $\bigcup_{n\geq 0} f^n(\operatorname{sing}(f^{-1}))$, where $\operatorname{sing}(f^{-1})$ denotes the set of singularities of the inverse of f. McMullen showed that the Julia set of an entire function f has Lebesgue measure zero if $\mathcal{P}(f)$ is a compact subset of $\mathcal{F}(f)$ and the density of $\mathcal{J}(f)$ in any large disk is bounded by a constant strictly less that one. Stallard extended the result to functions whose possibly unbounded postsingular set has positive distance to the Julia set. Jankowski and Zheng showed that one may allow for certain exceptions to this condition within a bounded subset of $\mathbb{C}$. We extend the result to functions which may have unbounded sequences of postsingular values whose distance to the Julia set tends to zero, provided that the density of the Julia set close to these values is not too large.

As an application, we consider the function f from Newton's method for finding the zeros of the function $g(z) = \int_0^z p(t)e^{q(t)}\,dt + c$ with polynomials p and q. We show that under certain assumptions on the zeros of g'' which are not zeros of g or g' – that is, the critical points of f which are not fixed points of f – the Julia set of f has Lebesgue measure zero. Together with a result of Bergweiler this implies that $f^n(z)$ converges to a zeros of g almost everywhere in $\mathbb{C}$ if this is the case for each zero z of g''.

Contrasting with the above results, McMullen showed that the Julia set of $\sin(az+b)$ for $a \neq 0$ has positive measure, and Schubert proved that the measure of the Fatou set of $\sin(z)$ is finite in any vertical strip of width 2π. A result of Sixsmith says that the Julia set of the function $f(z) = \sum_{k=1}^N a_k \exp(b_k z)$ has positive measure if the set of arguments of the b_k intersects, modulo 2π, every open interval of length π. We show that under similar assumptions on the numbers b_k, the Fatou set of the function $f(z) = \sum_{k=1}^N Q_k(z)\exp(b_k z^d + P_k(z))$, where P_k and Q_k are polynomials, $d \geq 3$ and $\deg(P_k) < d$, has finite measure. In fact, we even show that the Lebesgue measure of $\mathbb{C} \setminus (\mathcal{J}(f) \cap \mathcal{A}(f))$ is finite, where $\mathcal{A}(f)$ denotes the fast escaping set of f, consisting of those points where the iterates tend to infinity as fast as possible.

Zusammenfassung

Die Iterationstheorie meromorpher Funktionen, die ihren Ursprung in den Arbeiten von Fatou und Julia zur Iteration rationaler Funktionen um 1920 hat, ist heutzutage ein aktives Forschungsgebiet. Das wichtigste Konzept bei der Untersuchung des Iterationsverhaltens einer meromorphen Funktion f ist die Unterteilung der komplexen Ebene in zwei Mengen: die Fatoumenge $\mathcal{F}(f)$, in der alle Iterierten f^n definiert sind und eine normale Familie im Sinne von Montel bilden, und ihr Komplement, die Juliamenge $\mathcal{J}(f)$. In dieser Arbeit beschäftigen wir uns mit dem Lebesguemaß von Fatou- und Juliamengen und untersuchen dabei zwei gegensätzliche Phänomene. Zunächst betrachten wir meromorphe Funktionen, deren Juliamenge das Maß null hat, danach zeigen wir, dass die Fatoumenge gewisser ganzer Funktionen endliches Maß hat.

Die postsinguläre Menge $\mathcal{P}(f)$ einer meromorphen Funktion f ist der Abschluss der Menge $\bigcup_{n\geq 0} f^n(\operatorname{sing}(f^{-1}))$, wobei $\operatorname{sing}(f^{-1})$ die Menge der Singularitäten der Umkehrfunktion von f bezeichnet. Ein Resultat von McMullen besagt, dass die Juliamenge einer ganzen Funktion f das Lebesguemaß null hat, wenn $\mathcal{P}(f)$ eine kompakte Teilmenge von $\mathcal{F}(f)$ ist und die Dichte von $\mathcal{J}(f)$ in großen Kreisscheiben durch eine Konstante kleiner als eins beschränkt ist. Dieses Ergebnis wurde von Stallard auf Funktionen erweitert, deren möglicherweise unbeschränkte postsinguläre Menge positiven Abstand zur Juliamenge hat. Jankowski und Zheng zeigten, dass innerhalb einer beschränkten Teilmenge von $\mathbb{C}$ gewisse Ausnahmen zu dieser Bedingung zugelassen werden können. Wir erweitern das Resultat auf Funktionen, die unbeschränkte Folgen von postsingulären Werten besitzen dürfen, deren Abstand zur Juliamenge gegen null geht, sofern die Dichte der Juliamenge in der Nähe dieser Werte nicht zu groß ist.

Als Anwendung betrachten wir die Funktion f aus dem Newtonverfahren zur Bestimmung der Nullstellen der Funktion $g(z) = \int_0^z p(t)e^{q(t)}dt + c$ mit Polynomen p und q. Wir zeigen, dass unter gewissen Voraussetzungen an die Nullstellen von g'', die keine Nullstellen von g oder g' sind – das heißt, die kritischen Punkte von f, die keine Fixpunkte von f sind – die Juliamenge von f das Lebesguemaß null hat. Zusammen mit einem Resultat von Bergweiler folgt daraus, dass die Iterierten $f^n(z)$ fast überall in $\mathbb{C}$ gegen Nullstellen von g konvergieren, wenn dies für jede Nullstelle z von g'' der Fall ist.

Im Kontrast zu obigen Resultaten zeigte McMullen, dass die Juliamenge der Funktion $\sin(az+b)$ für $a \neq 0$ positives Maß hat, und Schubert bewies, dass das Maß der Fatoumenge von $\sin(z)$ eigeschränkt auf vertikale Streifen der Breite 2π endlich ist. Ein Resultat von Sixsmith besagt, dass die Juliamenge der Funktion $f(z) = \sum_{k=1}^N a_k \exp(b_k z)$ positives Maß hat, falls die Menge der Argumente der b_k modulo 2π jedes offene Intervall der Länge π schneidet. Wir zeigen, dass unter ähnlichen Voraussetzungen an die Zahlen b_k die Fatoumenge der Funktion $f(z) = \sum_{k=1}^N Q_k(z)\exp(b_k z^d + P_k(z))$ mit Polynomen P_k und Q_k, einer natürlichen Zahl $d \geq 3$ und $\deg(P_k) < d$ endliches Maß hat. Tatsächlich zeigen wir sogar, dass die Menge $\mathbb{C} \setminus (\mathcal{J}(f) \cap \mathcal{A}(f))$ endliches Maß hat, wobei $\mathcal{A}(f)$ die schnell entkommende Menge von f bezeichnet, in der die Iterierten von f so schnell wie möglich gegen unendlich streben.

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Contents

Chapter 1

Introduction

The foundations for the theory of complex dynamics were laid by Pierre Fatou [Fat19, Fat20a, Fat20b] and Gaston Julia [Jul18] who published long memoirs on iteration of rational functions between 1918 and 1920. They both divided the Riemann sphere in two parts: the set where the iterates behave stable, today called the Fatou set, and the set where chaotic behaviour occurs, today called the Julia set. Fatou [Fat26] later showed that many of the results obtained for rational functions carry over to transcendental entire functions, but also new phenomena arise. More recently, also iteration of transcendental meromorphic functions has been studied.

In order to describe the essential ideas of complex dynamics, let f be a rational function of degree at least two or a transcendental meromorphic function, and let f^n denote its nth iterate. Then a point z belongs to the *Fatou set* $\mathcal{F}(f)$ of f if all iterates f^n are defined and form a normal family in a neighbourhood of z; and the *Julia set* $\mathcal{J}(f)$ is the complement of $\mathcal{F}(f)$. Here, a family $\mathcal{G}$ of meromorphic functions is called *normal* if every sequence in $\mathcal{G}$ has a locally uniformly convergent subsequence. By its definition, $\mathcal{F}(f)$ is open and $\mathcal{J}(f)$ is closed. Moreover, $\mathcal{J}(f)$ is an infinite set, and either $\mathcal{F}(f) = \emptyset$ or $\mathcal{J}(f)$ has empty interior. An introduction to complex dynamics can be found in [Bea91, CG93, Mil06, Ste93] for rational functions; [MNTU00] for rational and entire functions; and [Ber93a] for transcendental meromorphic, including entire, functions.

There has been significant interest in the size of Fatou and Julia sets. We will concentrate on their Lebesgue measure here, but it should be mentioned that, for example, also the Hausdorff dimension of Julia sets has been extensively studied (see, e.g., [Gar78, McM87, Sta91, Sta94, Bis18]).

This thesis is divided into two parts. The first part concerns meromorphic functions whose Julia set has Lebesgue measure zero, in the second part we study a class of transcendental entire functions whose Fatou set has finite Lebesgue measure.

Julia sets of Lebesgue measure zero For a meromorphic function f, let $\operatorname{sing}(f^{-1})$ denote the set of finite singular values of f, that is, the closure of the set of critical and asymptotic values of f in $\mathbb{C}$ (see Definition 2.2.1). The *postsingular set* of f is defined as

$$\mathcal{P}(f) := \mathbb{C} \cap \overline{\bigcup_{n\geq 0} f^n(\operatorname{sing}(f^{-1}))}.$$

If f is a rational function whose postsingular set is a compact subset of its Fatou set, then the Lebesgue measure of its Julia set is zero. This follows from a theorem of Sullivan [Sul83, p. 745, see p. 742 for the definition of expanding] together with a result of Fatou [Fat20a, pp. 72f., see p. 60 for the definition of E_c and E_c']. The corresponding result is not true for transcendental functions. For example, if $f(z) = \sin(az)$ with $a \in \mathbb{C} \setminus \{0\}$ and $|a| < 1$, then the postsingular set of f is a compact subset of $\mathcal{F}(f)$ (see Example 3.3.1), and the Lebesgue measure of $\mathcal{J}(f)$ is positive [McM87, Theorem 1.1]. Nevertheless, one can show that the Julia set of a transcendental function has Lebesgue measure zero if, in addition to suitable assumptions on the postsingular set, the density of the Julia set in certain subsets of the complex plane is not too large. For $z_0 \in \mathbb{C}$ and $r > 0$, let $\mathcal{D}(z_0, r)$ be the open disk centred at z_0 with radius r, and let $\operatorname{meas}(\cdot)$ denote Lebesgue measure. Following McMullen [McM87], we call a measurable set $A \subset \mathbb{C}$ *thin at infinity* if there exist $R_0, \varepsilon_0 > 0$ such that for all $z \in \mathbb{C}$, we have

$$\operatorname{dens}(A, \mathcal{D}(z, R_0)) := \frac{\operatorname{meas}(A \cap \mathcal{D}(z, R_0))}{\operatorname{meas}(\mathcal{D}(z, R_0))} < 1 - \varepsilon_0.$$

McMullen [McM87, Proposition 7.3, see p. 337 for the definition of expanding] proved that if f is an entire function such that $\mathcal{P}(f)$ is a compact subset of $\mathcal{F}(f)$ and $\mathcal{J}(f)$ is thin at infinity, then $\mathcal{J}(f)$ has Lebesgue measure zero. This applies, for example, to $f(z) = \lambda e^z$ with $\lambda \in (0, 1/e)$ [McM87, Theorem 1.3]. Meromorphic functions whose postsingular set is a compact subset of its Fatou set are called *hyperbolic*. Iteration of hyperbolic meromorphic functions has been extensively studied (see, e.g., [Sta99, RS99, BFRG15, Zhe15, RGS17]).

Stallard [Sta90] extended McMullen's result to entire functions whose postsingular set has positive distance to the Julia set and whose Julia set is thin at infinity. This applies, for example, to $f(z) = z + 1 - e^z$. Meromorphic functions whose postsingular set has positive distance to the Julia set are sometimes called *topologically hyperbolic*, and were also considered in [MU10, BFJK20].

Jankowski ([Jan96, Satz 1],[Jan97, Theorem 3]) generalised Stallard's result by allowing that f is meromorphic and there are certain exceptions to the condition that the distance between $\mathcal{P}(f)$ and $\mathcal{J}(f)$ is positive. A further generalisation was later obtained by Zheng [Zhe02, Theorem 5] who proved that if f is a meromorphic function such that $\mathcal{P}(f) \cap \mathcal{J}(f)$ is a finite set, if there exists $R > 0$ such that $\mathcal{J}(f) \setminus \mathcal{D}(0, R)$ has positive distance to the postsingular set of f, and if $\mathcal{J}(f)$ is thin at infinity, then $\mathcal{J}(f)$ has Lebesgue measure zero.

Results on Julia sets of Lebesgue measure zero where the asssumption that the Julia set is thin at infinity is replaced by alternative assumptions are given in [EL92, Theorem 8] and [Zhe02, Theorems 3 and 4].

There are entire functions whose Julia set is thin at infinity and has positive measure. The Julia set of a polynomial is always bounded and thus thin at infinity, and Buff and Chéritat [BC12] showed that there are quadratic polynomials whose Julia set has positive measure. Also, an extension of an example of Eremenko and Lyubich [EL87, Example 4] yields that there are transcendental entire functions whose Julia set is thin at infinity and has positive measure (see Example 3.3.2).

We are able to further generalise the results of McMullen, Stallard, Jankowski and Zheng stated above, allowing for infinitely many singular values in the Julia set or an unbounded sequence of postsingular values whose distance to the Julia set tends to zero. In order to state this result, we call a set $A \subset \mathbb{C}$ *thin* at a point $z_0 \in \mathbb{C}$ if there exist $\delta_1, \varepsilon_1 > 0$ such that for all $z \in \overline{\mathcal{D}(z_0, \delta_1)}$, we have

$$\operatorname{dens}(A, \mathcal{D}(z, |z - z_0|)) < 1 - \varepsilon_1; \tag{1.0.1}$$

and we call A *uniformly thin* at a set $B \subset \mathbb{C}$ if there are $\delta_1, \varepsilon_1 > 0$ such that (1.0.1) holds for all $z_0 \in B$. Note that if, for some $R > 0$, the postsingular set of f has positive distance to $\mathcal{J}(f) \setminus \mathcal{D}(0, R)$, then $\mathcal{J}(f)$ is uniformly thin at $\mathcal{P}(f) \setminus \mathcal{D}(0, R')$ for all $R' > R$. We will prove the following result.

Theorem A. *Let f be a meromorphic function that is not constant and not a Möbius transformation. Suppose that there exists a compact set $\mathcal{P}_1 \subset \mathcal{P}(f)$ such that*

(i) $\mathcal{P}_1 \cap \mathcal{J}(f)$ *is a finite set;*

(ii) $\mathcal{J}(f)$ *is thin at infinity;*

(iii) $\mathcal{J}(f)$ *is uniformly thin at* $\mathcal{P}(f) \setminus \mathcal{P}_1$.

Then the Lebesgue measure of $\mathcal{J}(f)$ *is zero.*

Application to Newton's method As an application of Theorem A, we consider Newton's method for a certain class of entire functions. *Newton's method* for finding the zeros of a non-constant meromorphic function g consists of iterating the function

$$f(z) = z - \frac{g(z)}{g'(z)}.$$

We also call f the *Newton map* of g. One can show that for each zero z_0 of g, there is a connected component U of $\mathcal{F}(f)$ such that $f^n|_U \to z_0$ as $n \to \infty$. Jankowski [Jan96, §3] proved that if f is the function from Newton's method for

$$g(z) = r(z)e^{az} + b, \tag{1.0.2}$$

where r is a rational function with $r \not\equiv 0$ and $a, b \in \mathbb{C} \setminus \{0\}$, and if for each of the zeros $z_1, ..., z_N$ of g'' that are not zeros of g or g', the iterates $f^n(z_j)$ converge to a finite limit as $n \to \infty$, then the Julia set of f has Lebesgue measure zero.

We consider functions of the form

$$g(z) = \int_0^z p(t)e^{q(t)}\,dt + c, \tag{1.0.3}$$

where p, q are polynomials and $c \in \mathbb{C}$. Note that this includes the functions (1.0.2) if r is a polynomial.

We assume that g does not have the form

$$g(z) = \tilde{p}(z)e^{\tilde{q}(z)} \tag{1.0.4}$$

with polynomials $\tilde{p}$ and $\tilde{q}$. Then g has infinitely many zeros and f is transcendental. Newton's method for functions of the form (1.0.4) has been studied by Haruta [Har99].

We say that $z \in \mathbb{C}$ is *attracted by a periodic cycle C of f* if $\lim_{n\to\infty} \operatorname{dist}(f^n(z), C) = 0$, where $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{C}$.

Theorem B. *Let g be of the form* (1.0.3) *but not of the form* (1.0.4), *and let f be its Newton map. Denote the zeros of g'' which are not zeros of g or g' by $z_1, ..., z_N$. Suppose that for all $j \in \{1, ..., N\}$, either the point z_j is attracted by a periodic cycle of f or there exists $n \in \mathbb{N}$ with $f^n(z_j) = \infty$. Then the Lebesgue measure of $\mathcal{J}(f)$ is zero.*

The points $z_1, ..., z_N$ are precisely the critical points of f which are not superattracting fixed points, and they are called *free critical points*. Examples of functions satisfying the assumptions of Theorem B are given in Section 4.11. The essential difference between Theorem B and the aforementioned result by Jankowski about the functions (1.0.2) is that we allow the degree of q to be greater than one, which leads to an unbounded sequence of critical values of f whose distance to the Julia set tends to zero. Combining Theorem B and a result of Bergweiler [Ber93b, Theorem 3] (see Theorem 4.1.1) yields the following.

Corollary C. *Let g be of the form (1.0.3) but not of the form (1.0.4), and let f be its Newton map. Denote the zeros of g'' which are not zeros of g or g' by $z_1, ..., z_N$. If $f^n(z_j)$ converges to a finite limit for all $j \in \{1, ..., N\}$, then $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$.*

Fatou sets of finite Lebesgue measure We now turn to a contrasting situation. Let f denote a transcendental entire function. The *escaping set* $\mathcal{I}(f)$, consisting of those points where $f^n(z)$ tends to infinity, plays an important role in the dynamics of entire functions. Eremenko [Ere89] showed that $\mathcal{I}(f)$ is non-empty and $\mathcal{J}(f) = \partial\mathcal{I}(f)$. The *fast escaping set* $\mathcal{A}(f)$, introduced by Bergweiler and Hinkkanen [BH99], is a subset of the escaping set that roughly speaking consists of those points which tend to infinity as fast as possible under iteration. It is defined as the set of all $z \in \mathbb{C}$ such that, for some $l \in \mathbb{N}$,

$$|f^n(z)| \geq M^{n-l}(R, f) \text{ for } n > l.$$

Here, $M(r, f) = \max_{|z|=r} |f(z)|$ denotes the *maximum modulus*, $M^n(r, f)$ is its nth iterate with respect to r, and R is chosen such that $M(r, f) > r$ for $r \geq R$. The fast escaping set is always non-empty, and $\mathcal{J}(f) = \partial\mathcal{A}(f)$. See [RS12] for a detailed discussion of the fast escaping set.

McMullen [McM87, Theorem 1.1] showed that the Julia set of $f(z) = \sin(az + b)$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, has positive Lebesgue measure. His proof actually yields that $\mathcal{J}(f) \cap \mathcal{A}(f)$ has positive measure. Sixsmith [Six15, Theorem 1.1] proved that if $f(z) = \sum_{j=1}^{q} a_j \exp(\omega_q^j z)$, where $q \geq 2$, $a_j \in \mathbb{C} \setminus \{0\}$ and $\omega_q = \exp(2\pi i/q)$, then $\mathcal{J}(f) \cap \mathcal{A}(f)$ has positive measure. Sixsmith remarked without proof that his result remains true for

$$f(z) = \sum_{j=1}^{q} a_j \exp(b_j z), \tag{1.0.5}$$

where $q \geq 3$, $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $\arg(b_j) < \arg(b_{j+1}) < \arg(b_j) + \pi$ for $j \in \{1, ..., q-1\}$ and $\arg(b_q) > \arg(b_1) + \pi$, with the argument chosen in $[0, 2\pi)$. Bergweiler and Chyzhykov [BC16] showed that under suitable assumptions, the Julia set and escaping set of a transcendental entire function of completely regular growth have positive measure. These assumptions are satisfied for the functions (1.0.5). In fact, they are also satisfied if $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, ..., q-1\}$ or $\arg(b_q) = \arg(b_1) + \pi$, and if the a_k are polynomials instead of constants. Further classes of entire functions whose Julia set and (fast) escaping set have positive measure are presented in [AB12, Ber18].

For certain functions, there are stronger results in the sense that one can bound the size of the complement of the Julia or (fast) escaping set. Schubert [Sch08] used McMullen's methods to prove that for $f(z) = \sinh(z)$, the Lebesgue measure of $\mathcal{F}(f)$ and $\mathbb{C} \setminus \mathcal{I}(f)$ is finite in any horizontal strip of width 2π. In fact, his proof yields that $\mathcal{I}(f)$ may be replaced by the fast escaping set $\mathcal{A}(f)$ here. Zhang and Yang [ZY18] extended Schubert's result to functions of the form $P(e^z)/e^z$, where P is a polynomial of degree at least two satisfying $P(0) \neq 0$.

There seem to be no papers whose main objective is to show that the Lebesgue measure of the Fatou set or the complement of the (fast) escaping set of certain transcendental entire functions is finite. However, there are some results presented in papers mainly treating a different subject. Hemke [Hem05, Theorem 5.1] showed that if

$$f(z) = Q_1(z)e^{P(z)} + Q_2(z)e^{-P(z)}, \tag{1.0.6}$$

where P, Q_1, Q_2 are polynomials with $Q_1, Q_2 \not\equiv 0$ and $\deg(P) \geq 3$, then $\mathbb{C} \setminus \mathcal{I}(f)$ has finite Lebesgue measure. An example of such a function is given by $f(z) = \sin(z^3)$. A result of Bock [Boc96, Example 2] says that the Fatou set of $f(z) = \sin(\pi z)$ is empty, and $f^n(z)$ tends to infinity for almost all $z \in \mathbb{C}$.

This is different for $f(z) = \sin(z)$, which is conjugate to the function $\sinh(z)$ considered by Schubert, and $f(z) = \sin(z^2)$. For both functions, the Lebesgue measure of $\mathcal{F}(f)$ and $\mathbb{C} \setminus \mathcal{I}(f)$ is infinite (see Example 5.5.1).

We consider *exponential polynomials* of the form

$$f(z) = \sum_{j=1}^{N} Q_j(z) \exp(b_j z^d + P_j(z)), \tag{1.0.7}$$

where Q_j and P_j are polynomials with $Q_j \not\equiv 0$ and $\deg(P_j) < d$, and $b_j \in \mathbb{C} \setminus \{0\}$ are pairwise distinct numbers. We are able to prove that under certain assumptions, the Lebesgue measure of $\mathbb{C} \setminus (\mathcal{J}(f) \cap \mathcal{A}(f))$ is finite.

Theorem D. *Suppose f has the form* (1.0.7)*, where $d \geq 3$, and the numbers b_j satisfy* $\arg(b_j) \leq \arg(b_{j+1}) < \arg(b_j) + \pi$ *for all $j \in \{1, ..., N-1\}$ and* $\arg(b_N) > \arg(b_1) + \pi$, *with the argument chosen in* $[0, 2\pi)$. *Then the Lebesgue measure of* $\mathbb{C} \setminus (\mathcal{J}(f) \cap \mathcal{A}(f))$ *is finite.*

Note that the assumptions on the numbers b_j imply that $N \geq 3$. Recall that Sixsmith's result for the functions (1.0.5) remains true if $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, ..., q-1\}$ or $\arg(b_q) = \arg(b_1) + \pi$. This is not true in general for Theorem D. For example, the function $h(z) := (1/2)\exp(z^3 + iz) - (1/2)\exp(-z^3 + iz)$ has a superattracting fixed point at zero, and the corresponding attractive basin has infinite Lebesgue measure (see Section 5.5). However, if the polynomials P_j exhibit a certain structure, the conclusion of Theorem D remains true if $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, ..., N-1\}$ or $\arg(b_N) = \arg(b_1) + \pi$. For example, this is the case if the degree of $P_j - (b_j/b_{j+1})P_{j+1}$ or $P_1 - (b_1/b_N)P_N$, respectively, is at most $d - 3$ (see Theorem E). This is satisfied for the functions (1.0.6) considered by Hemke.

Outline In Chapter 2 we collect preliminary results required for the understanding of the main part of this thesis. In Chapter 3 we consider Julia sets of Lebesgue measure zero and prove Theorem A. As an application, we prove Theorem B in Chapter 4. Finally, in Chapter 5 we are concerned with Fatou and non-escaping sets of finite Lebesgue measure, and prove a generalisation of Theorem D.

Chapter 2

Preliminaries

This chapter contains notation and background material that will be needed in the main part of this thesis. For the most part, we only state the results, and refer to the literature for their proofs. The reader is expected to be familiar with basic definitions and results from analysis, in particular from function theory.

2.1 Notation

This section fixes some notation. The Riemann sphere is denoted by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we write $\mathcal{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for the open disk centred at z_0 with radius r. We use the notation $\operatorname{meas}(A)$ for the Lebesgue measure of $A \subset \mathbb{C}$ or, more generally, $A \subset \mathbb{R}^n$. For measurable subsets $A, B \subset \mathbb{C}$ with $\operatorname{meas}(B) \in (0, \infty)$, the *density* of A in B is defined as

$$\operatorname{dens}(A, B) := \frac{\operatorname{meas}(A \cap B)}{\operatorname{meas}(B)}.$$

Also, let $\operatorname{dist}(A, B) := \inf\{|z - w| : z \in A, w \in B\}$ denote the Euclidean distance of the sets $A, B \subset \mathbb{C}$. Analogously, we write $\operatorname{dist}(z, A)$ for the Euclidean distance of $z \in \mathbb{C}$ and $A \subset \mathbb{C}$. Moreover, the diameter of a set $A \subset \mathbb{C}$ is denoted by $\operatorname{diam}(A) := \sup\{|z - w| : z, w \in A\}$.

2.2 Injective holomorphic functions

In this section we collect injectivity criteria for holomorphic functions and state some properties of injective holomorphic maps.

Definition 2.2.1 (Singular value)**.** Let $D \subset \hat{\mathbb{C}}$ be a domain and $f : D \to \hat{\mathbb{C}}$ a meromorphic function.

- A point $w \in \hat{\mathbb{C}}$ is a *critical value* of f if there exists $z \in D$ such that $f(z) = w$ and f is not injective in any neighbourhood of z. For $z, w \in \mathbb{C}$, the latter is equivalent to $f'(z) = 0$.
- A point $w \in \hat{\mathbb{C}}$ is an *asymptotic value* of f if there exists a curve $\gamma : [0, \infty) \to D$ such that
$$\gamma(t) \to \partial D \text{ and } f(\gamma(t)) \to w \text{ as } t \to \infty. \tag{2.2.1}$$
- The set of *singular values* of f is defined as the closure of the set of all critical and asymptotic values of f.

Each singular value of a rational function is a critical value. A transcendental meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ may have asymptotic values. In this case, we have $\gamma(t) \to \infty$ in (2.2.1).

Lemma 2.2.2. *Let $D \subset \hat{\mathbb{C}}$ be a domain and $f : D \to \hat{\mathbb{C}}$ a meromorphic function. Let $W \subset \mathbb{C}$ be a simply connected domain that does not contain any singular value of f, and let V be a connected component of $f^{-1}(W)$. Then f maps V conformally onto W. In particular, for any $w \in W$ and $z \in f^{-1}(w)$, there exists a branch φ of f^{-1} defined in W with $\varphi(w) = z$.*

Since W does not contain any singular value, the map $f|_V : V \to W$ is a covering (cf. [GK86, Lemma 1.1]); and because W is simply connected, this implies that f is bijective (see [Jos02, Corollary 1.3.2]). An introduction to the theory of coverings can be found, for example, in [Jos02, §1.3].

Next, we give two injectivity criteria based on the behaviour of the function in its domain of definition. The following well-known criterion can be found, for example, in [Pom92, Proposition 1.10].

Lemma 2.2.3. *Suppose that the function f is holomorphic in a convex domain $D \subset \mathbb{C}$. If $\operatorname{Re} f'(z) > 0$ for all $z \in D$, then f is injective in D.*

We also require the following result.

Lemma 2.2.4. *Let $z_0 \in \mathbb{C}$ and $r > 0$, and let $f : \mathcal{D}(z_0, r) \to \mathbb{C}$ be a holomorphic function. Suppose that $f'(\zeta) \neq 0$ for all $\zeta \in \mathcal{D}(z_0, r)$ and*

$$\sup_{|\zeta - z_0| < r} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| < \frac{1}{r}.$$

Then f is injective in $\mathcal{D}(z_0, r)$.

This follows directly from Becker's univalence criterion (see, e.g., [Pom75, Theorem 6.7]). We sketch a more elementary proof of Lemma 2.2.4 based on Lemma 2.2.3.

Sketch of proof. We may assume without loss of generality that $f'(z_0) = 1$. Let ψ be the branch of $\log f'$ in $\mathcal{D}(z_0, r)$ satisfying $\psi(z_0) = 0$. Then, for all $z \in \mathcal{D}(z_0, r)$, we have

$$|\psi(z)| = \left| \int_{z_0}^{z} \frac{f''(\zeta)}{f'(\zeta)}\, d\zeta \right| \leq \sup_{|\zeta - z_0| < r} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \cdot |z - z_0| < \frac{1}{r} |z - z_0| < 1.$$

Thus $\arg(f'(z)) = \operatorname{Im} \psi(z) \in (-1, 1)$. In particular, $\operatorname{Re} f'(z) > 0$. By Lemma 2.2.3, f is injective in $\mathcal{D}(z_0, r)$. □

The next lemma can easily be deduced from the well-known Koebe 1/4-theorem and Koebe distortion theorem (see, e.g., [Con95, Theorems 7.8 and 7.9]).

Lemma 2.2.5 (Koebe)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$, and let $f : \mathcal{D}(z_0, r) \to \mathbb{C}$ be an injective holomorphic function. Then*

$$f(\mathcal{D}(z_0, r)) \supset \mathcal{D}\left(f(z_0), \frac{1}{4} |f'(z_0)| r \right).$$

Moreover, let $\rho \in (0, 1)$. Then for all $z \in \overline{\mathcal{D}(z_0, \rho r)}$, we have

$$\frac{|f'(z_0)|}{(1 + \rho)^2} \leq \frac{|f(z) - f(z_0)|}{|z - z_0|} \leq \frac{|f'(z_0)|}{(1 - \rho)^2}.$$

In particular,

$$f(\mathcal{D}(z_0, \rho r)) \subset \mathcal{D}\left(f(z_0), \frac{\rho}{(1-\rho)^2}|f'(z_0)|r\right).$$

Also,

$$\frac{\max_{|z-z_0|\le \rho r}|f'(z)|}{\min_{|z-z_0|\le \rho r}|f'(z)|} \le \left(\frac{1+\rho}{1-\rho}\right)^4.$$

More generally, if f is holomorphic in a domain $D \subset \mathbb{C}$ and K is a compact subset of D containing no critical points of f, then

$$L(f,K) := \frac{\max_{z\in K}|f'(z)|}{\min_{z\in K}|f'(z)|}$$

is called the *distortion* of f in K. From Lemma 2.2.5, one can deduce the following (see, e.g., [Con95, Theorem 7.16] for the first part).

Corollary 2.2.6. *Let $D \subset \mathbb{C}$ be a domain and let $K \subset D$ be compact. Then there exists a constant $C = C(D,K) \ge 1$ such that for any injective holomorphic function $f : D \to \mathbb{C}$, we have $L(f,K) \le C$.*

In fact, C can be chosen such that for any pair of injective holomorphic functions $g : D \to \mathbb{C}$ and $h : g(D) \to \mathbb{C}$, we have $L(h, g(K)) \le C$.

2.3 Lebesgue measure and holomorphic functions

This section concerns Lebesgue measure. The following lemma gives an estimate for the Lebesgue measure of a certain neighbourhood of a curve. We usually use the same notation for a curve and its trace.

Lemma 2.3.1. *Let $\gamma \subset \mathbb{C}$ be a curve of positive, finite Euclidean length, and let $s \in (0, \operatorname{length}(\gamma))$. Then*

$$\operatorname{meas}(\{z \in \mathbb{C} : \operatorname{dist}(z,\gamma) \le s\}) \le \frac{9\pi}{2} s \cdot \operatorname{length}(\gamma).$$

Proof. Let $L \in \mathbb{N}$ such that $L - 1 < \operatorname{length}(\gamma)/s \le L$. We divide γ into L subcurves $\gamma_1, ..., \gamma_L$ of length at most s. Then for $j \in \{1, ..., L\}$, there exists $a_j \in \mathbb{C}$ such that $\gamma_j \subset \overline{\mathcal{D}(a_j, s/2)}$. We have

$$\{z \in \mathbb{C} : \operatorname{dist}(z,\gamma_j) \le s\} \subset \overline{\mathcal{D}\left(a_j, \frac{3}{2}s\right)}.$$

Thus

$$\operatorname{meas}(\{z \in \mathbb{C} : \operatorname{dist}(z,\gamma_j) \le s\}) \le \frac{9\pi}{4}s^2.$$

Using that $\operatorname{length}(\gamma)/s + 1 \le 2\operatorname{length}(\gamma)/s$, we deduce

$$\operatorname{meas}(\{z \in \mathbb{C} : \operatorname{dist}(z,\gamma) \le s\}) \le L\frac{9\pi}{4}s^2 < \left(\frac{\operatorname{length}(\gamma)}{s} + 1\right)\frac{9\pi}{4}s^2 \le \frac{9\pi}{2}s \cdot \operatorname{length}(\gamma).$$

□

We require the following definition. Here, we use the notation $\mathcal{D}(z,r)$ also for balls in $\mathbb{R}^n$.

Definition 2.3.2 (Density point)**.** Let A be a measurable subset of $\mathbb{R}^n$. Then $z \in A$ is called a *(Lebesgue) density point* of A if

$$\lim_{r\to 0} \operatorname{dens}(A, \mathcal{D}(z,r)) = 1.$$

The following theorem can be found, for example, in [Mat95, Corollary 2.14].

Theorem 2.3.3 (Lebesgue density theorem)**.** *Let $A \subset \mathbb{R}^n$ be a set of positive Lebesgue measure. Then almost every $z \in A$ is a density point of A.*

The next lemma (see, e.g., [Pom92, p. 4]) is a direct consequence of the substitution rule in $\mathbb{R}^n$.

Lemma 2.3.4. *Let A be a measurable subset of $\mathbb{C}$. Suppose that the function f is holomorphic and injective in a neighbourhood of A. Then*

$$\operatorname{meas}(f(A)) = \int_A |f'(z)|^2 \, dx \, dy.$$

In particular,

$$\inf_{z\in A} |f'(z)|^2 \operatorname{meas}(A) \le \operatorname{meas}(f(A)) \le \sup_{z\in A} |f'(z)|^2 \operatorname{meas}(A).$$

Corollary 2.3.5. *Let $A \subset \mathbb{C}$ be a set of Lebesgue measure zero, and let f be a non-constant holomorphic function. Then* $\operatorname{meas}(f^{-1}(A)) = 0$.

Sketch of proof. Cover $f^{-1}(A) \setminus (f')^{-1}(0)$ by countably many disks D_n in each of which f is injective. If $\operatorname{meas}(f^{-1}(A) \cap D_n) > 0$ for some n, then Lemma 2.3.4 yields that $\operatorname{meas}(A \cap f(D_n)) > 0$, contradicting the assumption. So $\operatorname{meas}(f^{-1}(A) \cap D_n) = 0$ for all n, and hence $\operatorname{meas}(f^{-1}(A)) = 0$. □

2.4 Complex dynamics

This section contains an introduction to complex dynamics. For a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree at least two or a transcendental meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$, let

$$f^0(z) = z \quad \text{and} \quad f^n(z) = f(f^{n-1}(z)) \text{ for } n \in \mathbb{N};$$

that is, f^n is the nth iterate of f. Many important results about iteration of rational functions originate with Fatou [Fat19, Fat20a, Fat20b] and Julia [Jul18]. Fatou later also considered transcendental entire functions [Fat26]. An introduction to complex dynamics can be found, for example, in [Bea91, CG93, Mil06, Ste93] for rational functions; [MNTU00] for rational and entire functions; and [Ber93a] for transcendental meromorphic, including entire, functions.

We always implicitly assume that if f is a rational function, then the degree of f is at least two. By a meromorphic function we mean a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (of degree at least two) or a transcendental meromorphic (possibly entire) function $f : \mathbb{C} \to \hat{\mathbb{C}}$. Let $\mathcal{D}(f^n)$ denote the domain of definition of f^n. If f is rational, then $\mathcal{D}(f^n) = \hat{\mathbb{C}}$ for all $n \in \mathbb{N}$; and if f is transcendental entire, then $\mathcal{D}(f^n) = \mathbb{C}$ for all $n \in \mathbb{N}$. If f is a transcendental meromorphic function with at least one pole, then $\mathcal{D}(f) = \mathbb{C}$, and

$$\mathcal{D}(f^n) = \mathbb{C} \setminus \{z \in \mathbb{C} : f^j \text{ has a pole at } z \text{ for some } j < n\}$$

for $n \geq 2$. The term $f^n(A)$ for $A \subset \hat{\mathbb{C}}$ shall always be understood as $f^n(A \cap \mathcal{D}(f^n))$.

A key concept in complex dynamics is to divide $\mathcal{D}(f)$ in two parts, depending on whether the iterates show 'stable' (Fatou set) or 'chaotic' (Julia set) behaviour. In order to make this precise, we need the following definition.

Definition 2.4.1 (Normal family)**.** Let D be an open subset of $\hat{\mathbb{C}}$ and let $\mathcal{F}$ be a family of meromorphic functions from D to $\hat{\mathbb{C}}$. Then $\mathcal{F}$ is called *normal* if every sequence in $\mathcal{F}$ has a subsequence that converges locally uniformly in D with respect to the chordal metric.

An important criterion for normality is the following.

Theorem 2.4.2 (Montel)**.** *Let D be an open subset of $\hat{\mathbb{C}}$, and let $\mathcal{F}$ be a family of meromorphic functions from D to $\hat{\mathbb{C}}$. Suppose that there exist pairwise distinct numbers $a_1, a_2, a_3 \in \hat{\mathbb{C}}$ such that for all $f \in \mathcal{F}$, all $z \in D$ and all $j \in \{1, 2, 3\}$, we have $f(z) \neq a_j$. Then $\mathcal{F}$ is normal.*

This is due to Montel [Mon12, p. 497].

Definition 2.4.3 (Fatou and Julia set)**.** Let f be a meromorphic function.

- The *Fatou set* $\mathcal{F}(f)$ of f consists of those $z \in \mathcal{D}(f)$ such that all iterates f^n are defined and form a normal family in a neighbourhood of z.
- Its complement $\mathcal{J}(f) := \mathcal{D}(f) \setminus \mathcal{F}(f)$ is called the *Julia set* of f.

If f is rational or entire, then the assumption that all f^n are defined can be, and usually is, omitted from the definition of the Fatou set.

Lemma 2.4.4 (Properties of Fatou and Julia sets)**.** *Let f be a meromorphic function. Then*

(i) $\mathcal{F}(f)$ is open and $\mathcal{J}(f)$ is closed;

(ii) $\mathcal{J}(f)$ is an infinite set;

(iii) either $\mathcal{J}(f) = \mathcal{D}(f)$ or $\mathcal{J}(f)$ has empty interior;

(iv) $f^{-1}(\mathcal{F}(f)) = \mathcal{F}(f)$ and $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$;

(v) $f^{-1}(\mathcal{J}(f) \cup \{\infty\}) = \mathcal{J}(f)$ and $f(\mathcal{J}(f)) \subset \mathcal{J}(f) \cup \{\infty\}$.

For transcendental entire functions, the following set is also extensively studied.

Definition 2.4.5 (Escaping set)**.** For a transcendental entire function f, the set

$$\mathcal{I}(f) := \{z \in \mathbb{C} : \, f^n(z) \to \infty \text{ as } n \to \infty\}$$

is called the *escaping set* of f.

Lemma 2.4.6. *For every transcendental entire function f, we have $\mathcal{I}(f) \neq \emptyset$ and $\mathcal{J}(f) = \partial \mathcal{I}(f)$.*

This was proved by Eremenko [Ere89, pp. 339ff.]. The fast escaping set is a subset of the escaping set that roughly speaking consists of those points where the iterates tend to infinity as fast as possible.

Definition 2.4.7 (Fast escaping set)**.** Let f be a transcendental entire function. For $r > 0$, let $M(r,f) := \max_{|z|=r} |f(z)|$ denote the *maximum modulus* of f, and let $M^n(r,f)$ be its nth iterate with respect to r. Take $R > 0$ such that

$$M(r,f) > r \text{ for } r \geq R. \tag{2.4.1}$$

The *fast escaping set* $\mathcal{A}(f)$ of f consists of all $z \in \mathbb{C}$ such that there exists $l \in \mathbb{N}$ with

$$|f^n(z)| \geq M^{n-l}(R,f) \text{ for } n > l.$$

For every transcendental entire function, there exists $R > 0$ satisfying (2.4.1). The definition of $\mathcal{A}(f)$ is independent of the particular choice of R. The fast escaping set was introduced by Bergweiler and Hinkkanen [BH99], who also proved the following analogue of Lemma 2.4.6.

Lemma 2.4.8. *For every transcendental entire function f, we have $\mathcal{A}(f) \neq \emptyset$ and $\mathcal{J}(f) = \partial\mathcal{A}(f)$.*

A detailed discussion of the fast escaping set can be found in [RS12].

For a meromorphic function f and a set $A \subset \hat{\mathbb{C}}$, let

$$\mathcal{O}^+(A) := \bigcup_{n\geq 0} f^n(A)$$

and

$$\mathcal{O}^-(A) := \bigcup_{n\geq 1} f^{-n}(A)$$

denote the *forward orbit* and *backward orbit* of A, respectively. For $z \in \hat{\mathbb{C}}$, we also write $\mathcal{O}^\pm(z)$ instead of $\mathcal{O}^\pm(\{z\})$.

Definition 2.4.9 (Exceptional point)**.** A point $z \in \hat{\mathbb{C}}$ is called an *exceptional point* of the meromorphic function f if $\mathcal{O}^-(z)$ is finite.

One can show that any meromorphic function f has at most two exceptional points.

Lemma 2.4.10. *Let f be a meromorphic function, and let $z \in \mathcal{J}(f) \cup \{\infty\}$ be not an exceptional point of f. Then $\overline{\mathcal{O}^-(z)} = \mathcal{J}(f)$.*

This follows from Montel's theorem (Theorem 2.4.2). Montel's theorem also yields that for any open set U with $U \cap \mathcal{J}(f) \neq \emptyset$, the set $\hat{\mathbb{C}} \setminus \mathcal{O}^+(U)$ contains at most two points. In fact, we have the following stronger result.

Lemma 2.4.11. *Let f be a meromorphic function. Suppose that K is a compact subset of $\hat{\mathbb{C}}$ that does not contain any exceptional point of f, and let U be an open subset of $\mathcal{D}(f)$ with $U \cap \mathcal{J}(f) \neq \emptyset$. Then there exists $n_0 \in \mathbb{N}$ such that $K \subset f^n(U)$ for all $n \geq n_0$.*

Periodic points play an important role in iteration theory.

Definition 2.4.12 (Classification of periodic points)**.** Let f be a meromorphic function. A point $z_0 \in \mathcal{D}(f)$ is called a *periodic point* of period p of f if $f^p(z_0) = z_0$ and p is minimal with this property. Its multiplier is defined as

$$\lambda := \begin{cases} (f^p)'(z_0) & \text{if } z_0 \in \mathbb{C} \\ \left.\dfrac{\mathrm{d}}{\mathrm{d}z}\dfrac{1}{f^p(1/z)}\right|_{z=0} & \text{if } z_0 = \infty. \end{cases}$$

The periodic point z_0 is called

- *attracting* if $|\lambda| < 1$;
- *superattracting* if $\lambda = 0$;
- *rationally indifferent* if $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{Q}$;
- *irrationally indifferent* if $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$;
- *repelling* if $|\lambda| > 1$.

Attracting periodic points are in the Fatou set, whereas repelling and rationally indifferent periodic points are in the Julia set. For irrationally indifferent periodic points both cases occur. Moreover, one can show that the Julia set is the closure of the set of repelling periodic points. In fact, this was the original definition by Julia.

Definition 2.4.13 (Attracted by a cycle)**.** Let z_0 be a periodic point of period p of the meromorphic function f. We say that $z \in \mathcal{D}(f)$ is *attracted by the periodic cycle* $C := \{z_0, f(z_0), ..., f^{p-1}(z_0)\}$ if $\lim_{n\to\infty} \operatorname{dist}(f^n(z), C) = 0$.

Lemma 2.4.14. *Let C be a periodic cycle of the meromorphic function f. Suppose that $z \in \mathcal{J}(f)$ is attracted by C. Then there exists $n \in \mathbb{N}$ with $f^n(z) \in C$.*

Clearly, the assumptions imply that $C \subset \mathcal{J}(f)$. It is not difficult to see that the conclusion of Lemma 2.4.14 is true for repelling cycles. For rationally indifferent cycles, the result follows from the Leau-Fatou flower theorem ([Fat19, §10-11], see also [Mil06, §10]); and for irrationally indifferent cycles, it was shown by Pérez Marco [PM95]. The assumption that $z \in \mathcal{J}(f)$ is only needed for rationally indifferent cycles.

We now consider the behaviour of the iterates in connected components of the Fatou set, which are also called *Fatou components*.

Definition 2.4.15 (Periodic Fatou component/ wandering domain)**.** Let U be a Fatou component of the meromorphic function f. For $n \in \mathbb{N}$, denote by U_n the Fatou component with $f^n(U) \subset U_n$. Then U is called *periodic* if there exists $p \in \mathbb{N}$ with $U_p = U$. The smallest p with this property is called the period of U. We call U *preperiodic* if there exists $n \in \mathbb{N}$ such that U_n is periodic. If $U_m \neq U_n$ for all $m \neq n$, then U is called *wandering*.

Definition 2.4.16 (Classification of periodic Fatou components)**.** Let f be a meromorphic function, and let U be a periodic Fatou component of period p of f. Then f is called

(i) *immediate attractive basin* if U contains an attracting periodic point z_0 of period p of f and $f^{np}|_U \to z_0$ as $n \to \infty$;

(ii) *parabolic domain* if ∂U contains a periodic point z_0 of f such that $f^{np}|_U \to z_0$, with $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$;

(iii) *Baker domain* if there exists $z_0 \in \partial U$ such that $f^{np}|_U \to z_0$ as $n \to \infty$ and $f^p(z_0)$ is not defined;

(iv) *Siegel disk* if $f^p|_U$ is conformally conjugate to a rotation on a disk;

(v) *Herman ring* if $f^p|_U$ is conformally conjugate to a rotation on an annulus.

In case (i), we denote U by $\mathcal{A}^*(z_0)$, and we call

$$\mathcal{A}(z_0) := \{z \in \mathcal{D}(f) : f^{np}(z) \to z_0 \text{ as } n \to \infty\}$$

the *attractive basin* of z_0. So $\mathcal{A}^*(z_0)$ is the component of $\mathcal{A}(z_0)$ containing z_0.

Lemma 2.4.17 (Classification of periodic Fatou components)**.** *Every periodic Fatou component of a meromorphic function f is one of the five types introduced in Definition 2.4.16.*

The following lemma gives conditions under which a point cannot be in a periodic Fatou component.

Lemma 2.4.18. *Let f be a transcendental entire function and $z_0 \in \mathcal{I}(f)$. Let $z_n := f^n(z_0)$ for all $n \in \mathbb{N}$. Suppose that there exist $\lambda > 1$ and $N \geq 0$ such that*

$$f(z_n) \neq 0 \quad \text{and} \quad \left| z_n \frac{f'(z_n)}{f(z_n)} \right| \geq \lambda$$

for all $n \geq N$. Then z_0 is either in a multiply connected Fatou component or in the Julia set of f.

This is due to Sixsmith [Six15, Theorem 3.1]. The next lemma gives a sufficient condition for the existence of a Baker domain.

Lemma 2.4.19. *Let f be a transcendental meromorphic function. Suppose that there exist $d \in \mathbb{N}$, $r > 0$, $\varphi \in [0, 2\pi)$, $\delta > 0$, $\varepsilon \in (0, \pi/(2d))$ and $k \in \{0, ..., d-1\}$ such that*

$$f(z) = z + \frac{re^{i\varphi}}{z^{d-1}} + O\left(\frac{1}{|z|^{d-1+\delta}}\right)$$

as $z \to \infty$ uniformly in

$$\left\{ z \in \mathbb{C} : \left| \arg(z) - \frac{\varphi + 2k\pi}{d} \right| < \frac{\pi}{d} - \varepsilon \right\}.$$

Then f has an invariant Baker domain.

This is proved in [Fat19, §8, §11], see also [Hin92, Theorem 2].

If f has a superattracting fixed point z_0 of multiplicity k, then *Böttcher's functional equation*

$$\Psi(f(z)) = \Psi(z)^k$$

has a solution in a neighbourhood of z_0, with $\Psi(z_0) = 0$. Under suitable assumptions, Ψ extends to a conformal map between $\mathcal{A}^*(z_0)$ and $\mathcal{D}(0,1)$. This is made precise in the following theorem, which is stated in terms of $\Phi := \Psi^{-1}$.

Theorem 2.4.20. *Let f be a meromorphic function, and let z_0 be a superattracting fixed point of multiplicity k of f. Suppose that $\mathcal{A}^*(z_0)$ contains no critical point other than z_0 and no asymptotic value of f. Then there is a conformal map $\Phi : \mathcal{D}(0,1) \to \mathcal{A}^*(z_0)$ satisfying $\Phi(0) = z_0$ and*

$$f(\Phi(z)) = \Phi(z^k)$$

for all $z \in \mathcal{D}(0,1)$.

A proof of this theorem can be found, for example, in [Mil06, Theorem 9.3]. There the result is stated for rational functions, but the proof also works for meromorphic functions without asymptotic values in $\mathcal{A}^*(z_0)$.

Definition 2.4.21 (Postsingular set)**.** For a meromorphic function f, let $\operatorname{sing}(f^{-1})$ denote the set of singular values of f in $\mathcal{D}(f)$. The *postsingular set* $\mathcal{P}(f)$ of f is defined as

$$\mathcal{P}(f) := \mathbb{C} \cap \overline{\bigcup_{n \geq 0} f^n(\operatorname{sing}(f^{-1}))}.$$

One can show (see, e.g., [Bak70, Lemma 2]) that

$$\mathcal{P}(f) = \mathbb{C} \cap \overline{\bigcup_{n\geq 1} \operatorname{sing}(f^{-n})}.$$

There is a close connection between singular values and Fatou components.

Lemma 2.4.22. *Let f be a meromorphic function.*

(i) Every cycle of immediate attractive basins or parabolic domains of f contains a singular value of f.

(ii) The boundary in $\mathbb{C}$ of any cycle of Siegel disks or Herman rings of f is contained in the postsingular set of f.

2.5 Newton's method

Newton's method for finding the zeros of a meromorphic function g consists of iterating the function

$$f(z) = z - \frac{g(z)}{g'(z)}.$$

We also call f the *Newton map* of g. The function f is meromorphic, and we have

$$f'(z) = \frac{g(z)g''(z)}{g'(z)^2}.$$

Let z_0 be a zero of multiplicity m of g. Then $f(z_0) = z_0$ and

$$f'(z_0) = \frac{m-1}{m}.$$

In particular, z_0 is an attracting fixed point of f, and if z_0 is a simple zero of g, then it is even a superattracting fixed point of f. Thus there exists a subset of the Fatou set of f, namely the attractive basin $\mathcal{A}(z_0)$, where $f^n(z) \to z_0$ as $n \to \infty$.

2.6 Real estimates

This section contains two estimates that we will need in the main part of this thesis. For $\alpha > 0$, consider the function

$$E_\alpha : [0,\infty) \to [0,\infty),\ E_\alpha(x) = \exp(x^\alpha).$$

Lemma 2.6.1. *Let $\beta > \alpha > 0$. Then there exists $x_0 > 0$ such that*

$$E_\alpha^k(x) \geq E_\beta^{k-2}(x)$$

for all $k \geq 4$ and all $x \geq x_0$.

This is proved in [Ber18, Lemma 2.1].

Lemma 2.6.2. *For all $n_0 \in \mathbb{N}$, we have*

$$\sum_{k=n_0}^{\infty} \frac{1}{k^2} \leq \frac{2}{n_0}.$$

Proof. We have

$$\sum_{k=n_0}^{\infty} \frac{1}{k^2} \leq \frac{1}{n_0^2} + \sum_{k=n_0+1}^{\infty} \int_{k-1}^{k} \frac{1}{t^2}\, dt = \frac{1}{n_0^2} + \int_{n_0}^{\infty} \frac{1}{t^2}\, dt = \frac{1}{n_0^2} + \frac{1}{n_0} \leq \frac{2}{n_0}. \qquad \square$$

Chapter 3

Julia sets of Lebesgue measure zero

This chapter concerns Theorem A. We recall its statement in Section 3.1, before we prove it in Section 3.2. Finally, in Section 3.3 we briefly discuss two known examples of functions whose Julia sets have positive Lebesgue measure, and which satisfy part of the assumptions of Theorem A.

3.1 Main result

In the statement of Theorem A, we use the following terminology.

Definition 3.1.1. A measurable set $A \subset \mathbb{C}$ is called

(i) *thin at infinity* if there exist $R_0, \varepsilon_0 > 0$ such that for all $z \in \mathbb{C}$, we have

$$\operatorname{dens}(A, \mathcal{D}(z, R_0)) < 1 - \varepsilon_0;$$

(ii) *thin at* $z_0 \in \mathbb{C}$ if there exist $\delta_1, \varepsilon_1 > 0$ such that for all $z \in \overline{\mathcal{D}(z_0, \delta_1)}$, we have

$$\operatorname{dens}(\mathcal{A}, \mathcal{D}(z, |z - z_0|)) < 1 - \varepsilon_1; \tag{3.1.1}$$

(iii) *uniformly thin at* $B \subset \mathbb{C}$ if there are $\delta_1, \varepsilon_1 > 0$ such that (3.1.1) holds for all $z_0 \in B$.

Part (i) was introduced by McMullen [McM87], parts (ii) and (iii) are new. The main objective of this chapter is to prove the following result.

Theorem A. *Let f be a meromorphic function. Suppose that there exists a compact set $\mathcal{P}_1 \subset \mathcal{P}(f)$ such that*

(i) $\mathcal{P}_1 \cap \mathcal{J}(f)$ is a finite set;

(ii) $\mathcal{J}(f)$ is thin at ∞;

(iii) $\mathcal{J}(f)$ is uniformly thin at $\mathcal{P}(f) \setminus \mathcal{P}_1$.

Then the Lebesgue measure of $\mathcal{J}(f)$ is zero.

3.2 Proof of Theorem A

In Lemma 3.2.1 we state conditions under which a point in $\mathcal{J}(f)$ is not a density point of $\mathcal{J}(f)$. Afterwards, we use Lemma 3.2.1 and the Lebesgue density theorem to prove Theorem A.

Lemma 3.2.1. *Let f be a meromorphic function and $z \in \mathcal{J}(f) \setminus \mathcal{O}^-(\mathcal{P}(f) \cup \{\infty\})$. Suppose that there exist sequences (n_k) in $\mathbb{N}$ with $\lim_{k\to\infty} n_k = \infty$ and (r_k) in $(0, \infty)$ satisfying the following conditions:*

(i) $\mathcal{D}(f^{n_k}(z), r_k) \cap \mathcal{P}(f) = \emptyset$ for all $k \in \mathbb{N}$;

(ii) there exists $\varepsilon > 0$ such that $\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(f^{n_k}(z), r_k)) \geq \varepsilon$ for all $k \in \mathbb{N}$.

Then z is not a density point of $\mathcal{J}(f)$.

Proof. Using methods of McMullen [McM87], Stallard [Sta90] and Zheng [Zhe02], we relate the density of $\mathcal{F}(f)$ in small neighbourhoods of z to the density of $\mathcal{F}(f)$ in the disks $\mathcal{D}(f^{n_k}(z), r_k)$. See Figure 3.1 for an illustration of the approach.

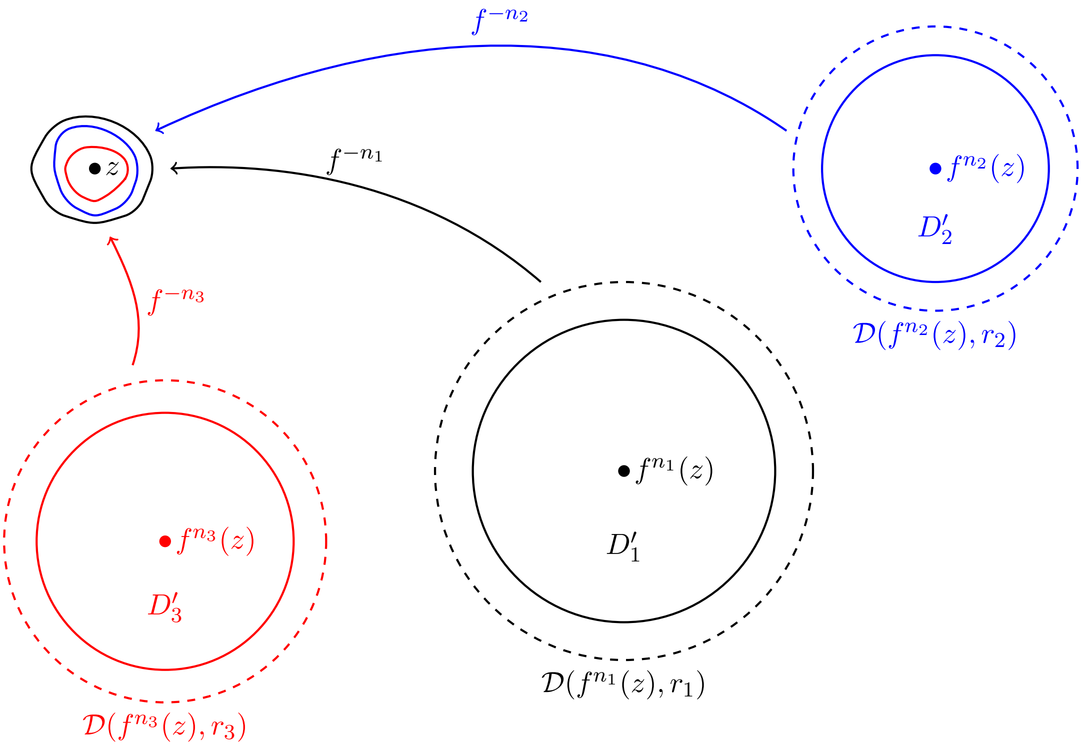


Figure 3.1: An illustration of the proof of Lemma 3.2.1. Suitable branches of f^{-n_k} map the disks D'_k with bounded distortion onto neighbourhoods of z. The diameter of these neighbourhoods tends to zero as $k \to \infty$.

Set

$$\omega := \sqrt{1 - \frac{\varepsilon}{2}}.$$

For $k \in \mathbb{N}$, let

$$z_k := f^{n_k}(z), \qquad D_k := \mathcal{D}(z_k, r_k) \qquad \text{and } D'_k := \mathcal{D}\left(z_k, \omega r_k\right).$$

Since $D_k \cap \mathcal{P}(f) = \emptyset$, there exists a branch φ_k of f^{-n_k} defined in D_k with $\varphi_k(z_k) = z$. By Koebe's theorems (Lemma 2.2.5),

$$\mathcal{D}\left(z, \frac{\omega}{4} r_k |\varphi_k'(z_k)|\right) \subset \varphi_k(D_k') \subset \mathcal{D}\left(z, \frac{\omega}{(1-\omega)^2} r_k |\varphi_k'(z_k)|\right). \tag{3.2.1}$$

We proceed in two steps.

1st step: We claim that

$$\lim_{k\to\infty} \left|\varphi_k'(z_k)\right| r_k = 0. \tag{3.2.2}$$

Assume that (3.2.2) does not hold. Then there exists $\delta > 0$ such that for infinitely many k, we have $\mathcal{D}(z,\delta) \subset \varphi_k(D_k')$ and hence $f^{n_k}(\mathcal{D}(z,\delta)) \subset D_k'$. Fix $v \in \mathcal{P}(f)$ and let K be a set of the form $K := \{z \in \mathbb{C} : |z - v| = \rho\}$, where $\rho > 0$ is chosen such that K does not contain any exceptional point of f. Then by Lemma 2.4.11, $K \subset f^{n_k}(\mathcal{D}(z,\delta)) \subset D_k' \subset D_k$ for all large k. But this implies $v \in D_k$, contradicting (i). So (3.2.2) is proved.

2nd step: It remains to show that

$$\limsup_{r\to 0} \operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z,r)) > 0; \tag{3.2.3}$$

that is, z is not a density point of $\mathcal{J}(f)$. By Lemma 2.3.4 and the definition of the density,

$$\begin{aligned}
\operatorname{dens}(\mathcal{F}(f), \varphi_k(D_k')) &\geq \frac{\inf_{\zeta\in D_k'} |\varphi_k'(\zeta)|^2}{\sup_{\zeta\in D_k'} |\varphi_k'(\zeta)|^2} \operatorname{dens}(\mathcal{F}(f), D_k') \\
&= \left(\frac{\inf_{\zeta\in D_k'} |\varphi_k'(\zeta)|}{\sup_{\zeta\in D_k'} |\varphi_k'(\zeta)|}\right)^2 \frac{\operatorname{meas}(D_k' \cap \mathcal{F}(f))}{\operatorname{meas} D_k'} \\
&\geq \left(\frac{\inf_{\zeta\in D_k'} |\varphi_k'(\zeta)|}{\sup_{\zeta\in D_k'} |\varphi_k'(\zeta)|}\right)^2 \cdot \frac{\operatorname{meas}(D_k \cap \mathcal{F}(f)) - \operatorname{meas}(D_k \setminus D_k')}{\operatorname{meas} D_k}.
\end{aligned}$$

Hence, by the Koebe distortion theorem and (ii),

$$\operatorname{dens}(\mathcal{F}(f), \varphi_k(D_k')) \geq \left(\frac{1-\omega}{1+\omega}\right)^8 \cdot \left(\varepsilon - \frac{\pi r_k^2 - \pi r_k^2 \omega^2}{\pi r_k^2}\right) = \left(\frac{1-\omega}{1+\omega}\right)^8 \cdot \frac{\varepsilon}{2}. \tag{3.2.4}$$

By (3.2.1) and (3.2.4),

$$\begin{aligned}
&\operatorname{dens}\left(\mathcal{F}(f), \mathcal{D}\left(z, \frac{\omega}{(1-\omega)^2} |\varphi_k'(z_k)| r_k\right)\right) \\
&\geq \operatorname{dens}(\mathcal{F}(f), \varphi_k(D_k')) \cdot \operatorname{dens}\left(\varphi_k(D_k'), \mathcal{D}\left(z, \frac{\omega}{(1-\omega)^2} |\varphi_k'(z_k)| r_k\right)\right) \\
&\geq \left(\frac{1-\omega}{1+\omega}\right)^8 \frac{\varepsilon}{2} \cdot \frac{1}{16} (1-\omega)^4.
\end{aligned}$$

By (3.2.2), this proves (3.2.3). □

Proof of Theorem A. Set $\mathcal{P}_{\mathcal{J}} := \mathcal{P}(f) \cap \mathcal{J}(f)$. By the assumptions, there is a compact set $\mathcal{P}_1 \subset \mathcal{P}(f)$ such that the set $\mathcal{P}_{\mathcal{J},1} := \mathcal{P}_1 \cap \mathcal{P}_{\mathcal{J}}$ is finite and $\mathcal{J}(f)$ is uniformly thin at $\mathcal{P}(f) \setminus \mathcal{P}_1$. Hence there are $\delta_1, \varepsilon_1 > 0$ such that for all $v \in \mathcal{P}(f) \setminus \mathcal{P}_1$ and all $\zeta \in \overline{\mathcal{D}(v, \delta_1)}$, we have

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(\zeta, |\zeta - v|)) > \varepsilon_1. \tag{3.2.5}$$

We proceed in three steps. First, we prove that $\mathcal{P}_{\mathcal{J}}$ has Lebesgue measure zero, and conclude that the same is true for $\mathcal{O}^-(\mathcal{P}_{\mathcal{J}} \cup \{\infty\})$. Then we show that every $z \in \mathcal{J}(f) \setminus \mathcal{O}^-(\mathcal{P}_{\mathcal{J}} \cup \{\infty\})$ satisfies $\limsup_{n\to\infty} \operatorname{dist}(f^n(z), \mathcal{P}_{\mathcal{J},1}) > 0$. Finally, we prove that $\mathcal{J}(f) \setminus \mathcal{O}^-(\mathcal{P}_{\mathcal{J}} \cup \{\infty\})$ has measure zero.

1st step: First, we show that $\operatorname{meas}(\mathcal{P}_{\mathcal{J}}) = 0$. Let $z \in \mathcal{P}_{\mathcal{J},2} := \mathcal{P}_{\mathcal{J}} \setminus \mathcal{P}_1$ and $r \in (0, 2\delta_1)$. Then $\mathcal{D}(z + r/2, r/2) \subset \mathcal{D}(z, r)$ and $\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z + r/2, r/2)) > \varepsilon_1$. Thus

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z,r)) \geq \operatorname{dens}\left(\mathcal{F}(f), \mathcal{D}\left(z + \frac{r}{2}, \frac{r}{2}\right)\right) \cdot \operatorname{dens}\left(\mathcal{D}\left(z + \frac{r}{2}, \frac{r}{2}\right), \mathcal{D}(z,r)\right) > \frac{\varepsilon_1}{4}.$$

Hence z is not a density point of $\mathcal{J}(f)$. By the Lebesgue density theorem (Theorem 2.3.3), the Lebesgue measure of $\mathcal{P}_{\mathcal{J},2}$ is zero. Since $\mathcal{P}_{\mathcal{J},1}$ is finite, this yields that the measure of $\mathcal{P}_{\mathcal{J}}$ and hence also the measure of $\mathcal{O}^-(\mathcal{P}_{\mathcal{J}})$ is zero. Because $\mathcal{O}^-(\infty)$ is countable, we deduce that $\mathcal{O}^-(\mathcal{P}_{\mathcal{J}} \cup \{\infty\})$ has zero measure.

2nd step: Using arguments of [Jan97], we show that every $z \in \mathcal{J}(f) \setminus \mathcal{O}^-(\mathcal{P}(f) \cup \{\infty\})$ satisfies

$$\limsup_{n\to\infty} \operatorname{dist}(f^n(z), \mathcal{P}_{\mathcal{J},1}) > 0. \tag{3.2.6}$$

To this end, suppose that $\lim_{n\to\infty} \operatorname{dist}(f^n(z), \mathcal{P}_{\mathcal{J},1}) = 0$. We show that then $z \in \mathcal{O}^-(\mathcal{P}_{\mathcal{J},1})$, a contradiction to the assumption.

Because $\mathcal{P}_{\mathcal{J},1}$ is finite, there exists a subsequence $(f^{m_k}(z))$ that converges to some $w \in \mathcal{P}_{\mathcal{J},1}$. For all $j \in \mathbb{N}$, we have $f^j(w) = \lim_{k\to\infty} f^{m_k+j}(z) \in \mathcal{P}_{\mathcal{J},1}$. Thus there exists $l \in \mathbb{N}$ such that $f^l(w)$ is periodic. Assume without loss of generality that $l = 0$; that is, there is $p \in \mathbb{N}$ with $f^p(w) = w$.

Let $\alpha > 0$ such that the disks $\overline{\mathcal{D}(\zeta, \alpha)}$ for $\zeta \in \mathcal{P}_{\mathcal{J},1}$ are pairwise disjoint, and let $\beta \in (0, \alpha)$ so that $f(\mathcal{D}(f^j(w), \beta)) \subset \mathcal{D}(f^{j+1}(w), \alpha)$ for all $j \in \{0, ..., p-1\}$. Then by periodicity, this is true for all $j \geq 0$. If k is sufficiently large, then $f^{m_k}(z) \in \mathcal{D}(w, \beta)$, and $\operatorname{dist}(f^{m_k+j}(z), \mathcal{P}_{\mathcal{J},1}) < \beta$ for all $j \geq 0$. Then $f^{m_k+1}(z) \in \mathcal{D}(f(w), \alpha)$; and since the disks $\overline{\mathcal{D}(\zeta, \alpha)}$ for $\zeta \in \mathcal{P}_{\mathcal{J},1}$ are disjoint, we have $|f^{m_k+1}(z) - \zeta| > \alpha > \beta$ for all $\zeta \in \mathcal{P}_{\mathcal{J},1} \setminus \{f(w)\}$. Thus $f^{m_k+1}(z) \in \mathcal{D}(f(w), \beta)$. Inductively we deduce that $f^{m_k+j}(z) \in \mathcal{D}(f^j(w), \beta)$ for all $j \in \mathbb{N}$. Hence $f^n(z)$ is attracted by the cycle $\{w, f(w), ..., f^{p-1}(w)\}$. By Lemma 2.4.14, z is eventually mapped to this cycle, so $z \in \mathcal{O}^-(\mathcal{P}_{\mathcal{J},1})$.

3rd step: Let $z \in \mathcal{J}(f) \setminus \mathcal{O}^-(\mathcal{P}(f) \cup \{\infty\})$. We prove that z satisfies the assumptions of Lemma 3.2.1, and hence is not a density point of $\mathcal{J}(f)$. By (3.2.6), there exist $\eta > 0$ and a subsequence $(f^{n_k}(z))$ such that

$$\operatorname{dist}(f^{n_k}(z), \mathcal{P}_{\mathcal{J},1}) > \eta \tag{3.2.7}$$

for all $k \in \mathbb{N}$. Set

$$d_k := \operatorname{dist}(f^{n_k}(z), \mathcal{P}(f)),$$

and take $z_k \in \mathcal{P}(f)$ such that

$$|f^{n_k}(z) - z_k| = d_k.$$

Since $\mathcal{J}(f)$ is thin at infinity, there are R_0, $\varepsilon_0 > 0$ such that for all $v \in \mathbb{C}$, we have

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(v, R_0)) > \varepsilon_0.$$

Following [Sta90], we distinguish several cases depending on the behaviour of d_k and z_k as $k \to \infty$.

1st case: $d_k \geq R_0$ for infinitely many k. Taking a subsequence if necessary, we can assume that $d_k \geq R_0$ for all $k \in \mathbb{N}$. Then $\mathcal{D}(f^{n_k}(z), R_0) \cap \mathcal{P}(f) = \emptyset$ and

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(f^{n_k}(z), R_0)) > \varepsilon_0.$$

By Lemma 3.2.1, z is not a density point of $\mathcal{J}(f)$.

2nd case: $d_k \leq \delta_1$ and $z_k \in \mathcal{P}(f) \setminus \mathcal{P}_1$ for infinitely many k, without loss of generality for all $k \in \mathbb{N}$. Then by (3.2.5),

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(f^{n_k}(z), d_k)) > \varepsilon_1.$$

By Lemma 3.2.1, z is not a density point of $\mathcal{J}(f)$.

3rd case: $d_k \in (\delta_1, R_0)$ and $z_k \in \mathcal{P}(f) \setminus \mathcal{P}_1$ for infinitely many k, without loss of generality for all $k \in \mathbb{N}$. Let

$$w_k := z_k + \frac{\delta_1}{d_k}(f^{n_k}(z) - z_k).$$

Then

$$|f^{n_k}(z) - w_k| = \left(1 - \frac{\delta_1}{d_k}\right) |f^{n_k}(z) - z_k| = d_k - \delta_1,$$

and hence

$$\mathcal{D}(w_k, \delta_1) \subset \mathcal{D}(f^{n_k}(z), d_k).$$

Also, $|w_k - z_k| = \delta_1$. By the assumptions, we obtain

$$\begin{aligned}\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(f^{n_k}(z), d_k)) &\geq \operatorname{dens}(\mathcal{D}(w_k, \delta_1), \mathcal{D}(f^{n_k}(z), d_k)) \cdot \operatorname{dens}(\mathcal{F}(f), \mathcal{D}(w_k, \delta_1)) \\ &\geq \frac{\delta_1^2}{d_k^2}\varepsilon_1 \geq \frac{\delta_1^2}{R_0^2}\varepsilon_1.\end{aligned}$$

By Lemma 3.2.1, z is not a density point of $\mathcal{J}(f)$.

4th case: $d_k < R_0$ and $z_k \in \mathcal{P}_1$ for infinitely many k, without loss of generality for all $k \in \mathbb{N}$. Then $(f^{n_k}(z))$ is bounded; and taking a subsequence if necessary, we can assume that $f^{n_k}(z)$ converges to some $w \in \mathbb{C}$. Let

$$\nu := \min\left\{\frac{\eta}{2}, \operatorname{dist}\left(\mathcal{J}(f), \mathcal{P}_1 \setminus \bigcup_{\zeta \in \mathcal{P}_{\mathcal{J},1}} \mathcal{D}\left(\zeta, \frac{\eta}{2}\right)\right)\right\} > 0.$$

By (3.2.7), we have $d_k \geq \nu$ and hence $\mathcal{D}(f^{n_k}(z), \nu) \cap \mathcal{P}(f) = \emptyset$ for all $k \in \mathbb{N}$. For large k, we have $\mathcal{D}(w, \nu/2) \subset \mathcal{D}(f^{n_k}(z), \nu)$. Thus

$$\begin{aligned}\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(f^{n_k}(z), \nu)) &\geq \operatorname{dens}\left(\mathcal{D}\left(w, \frac{\nu}{2}\right), \mathcal{D}(f^{n_k}(z), \nu)\right) \cdot \operatorname{dens}\left(\mathcal{F}(f), \mathcal{D}\left(w, \frac{\nu}{2}\right)\right) \\ &= \frac{1}{4}\operatorname{dens}\left(\mathcal{F}(f), \mathcal{D}\left(w, \frac{\nu}{2}\right)\right) > 0.\end{aligned}$$

By Lemma 3.2.1, z is not a density point of $\mathcal{J}(f)$.

Altogether, it follows that the set of density points of $\mathcal{J}(f)$ has Lebesgue measure zero. The Lebesgue density theorem yields that $\mathcal{J}(f)$ has Lebesgue measure zero. $\square$

Remark 3.2.2. The fourth case is related to a theorem of Bock [Boc96] which says that for any entire function f with $\mathcal{J}(f) \neq \mathbb{C}$, we have $\lim_{n\to\infty} \operatorname{dist}_\chi(f^n(z), \mathcal{P}(f) \cup \{\infty\}) = 0$. Here, dist_χ denotes the distance with respect to the chordal metric in $\hat{\mathbb{C}}$.

3.3 Counterexamples

In the previous part of this chapter we have seen that if f is a meromorphic function whose Julia set is uniformly thin at the postsingular set and thin at infinity, then the Lebesgue measure of $\mathcal{J}(f)$ is zero. In this section we discuss two examples, due to [McM87] and [EL87], respectively, which show that there are entire functions f with $\operatorname{dist}(\mathcal{P}(f), \mathcal{J}(f)) > 0$ and Julia sets of positive measure as well as entire functions whose Julia sets are thin at infinity and have positive measure.

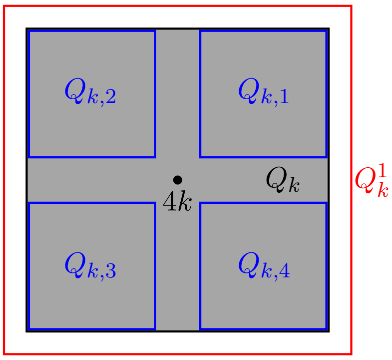


Figure 3.2: An illustration of the squares Q_k, Q_k^1 and $Q_{k,j}$.

Example 3.3.1. For $a \in \mathbb{C} \setminus \{0\}$, let $f_a(z) := \sin(az)$. If $|a| < 1$, then $\mathcal{P}(f_a)$ is a compact subset of $\mathcal{F}(f_a)$. However, McMullen [McM87, Theorem 1.1] showed that $\mathcal{J}(f_a)$ has positive measure for any $a \in \mathbb{C} \setminus \{0\}$.

To see that for $|a| < 1$ the postsingular set of f_a is a compact subset of $\mathcal{F}(f_a)$, note that $\operatorname{sing}(f_a^{-1}) = \{-1, 1\}$. Also, $f_a(0) = 0$ and $f_a'(0) = a$. So f_a has an attracting fixed point at zero; and by Lemma 2.4.22, the corresponding attractive basin $\mathcal{A}(0)$ contains a singular value of f_a. Since $f_a^n(-z) = -f_a^n(z)$, both singular values lie in $\mathcal{A}(0)$, and hence $\mathcal{P}(f_a)$ is a compact subset of $\mathcal{A}(0)$.

Example 3.3.2. There is a transcendental entire function whose Julia set is thin at infinity and has positive measure.

This slightly extends an example of Eremenko and Lyubich [EL87, Example 4]. They construct a transcendental entire function whose Julia set has positive measure using the following approximation result [EL87, Main Lemma].

Lemma 3.3.3. *Let (G_k) be a sequence of pairwise disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C} \setminus G_k$ is connected for all $k \in \mathbb{N}$ and $\min_{z \in G_k} |z| \to \infty$ as $k \to \infty$. Let $z_k \in G_k$ and $\varepsilon_k > 0$ for all $k \in \mathbb{N}$. Then for any function Φ holomorphic in a neighbourhood of $\bigcup_{k \geq 1} G_k$, there exists an entire function f such that for all $k \in \mathbb{N}$, we have $f(z_k) = \Phi(z_k)$, $f'(z_k) = \Phi'(z_k)$ and*

$$\max_{z \in G_k} |f(z) - \Phi(z)| < \varepsilon_k.$$

Next, we outline the construction of an entire function f whose Julia set has positive measure given in [EL87, Example 4]. Let (ε_k) be a sequence in $(0, \infty)$ such that $\sum_{k=0}^{\infty} \varepsilon_k < 1/36$. For $k \geq 0$, define squares

$$Q_k := \{z \in \mathbb{C} : |\operatorname{Re} z - 4k| < 1, |\operatorname{Im} z| < 1\}$$

and

$$Q_k^1 := \{z \in \mathbb{C} : |\operatorname{Re} z - 4k| < 1 + \varepsilon_k, |\operatorname{Im} z| < 1 + \varepsilon_k\}.$$

Then the set

$$Q_k \setminus (\{z \in \mathbb{C} : |\operatorname{Re} z - 4k| < \varepsilon_k\} \cup \{z \in \mathbb{C} : |\operatorname{Im} z| < \varepsilon_k\})$$

is the union of four squares, which we denote by $Q_{k,1}, \ldots, Q_{k,4}$. See Figure 3.2 for an illustration of these sets. Consider affine surjective maps $\Phi_{k,j} : Q_{k,j} \to Q_{k+1}^1$.

Eremenko and Lyubich construct an entire function f such that for all $k \geq 0$ and all $j \in \{1, \ldots, 4\}$,

(i) $\sup_{z \in Q_{k,j}} |f(z) - \Phi_{k,j}(z)| < \varepsilon_{k+1}$;

(ii) f is injective in $Q_{k,j}$ and $|f'(z)| > 2$ for all $z \in Q_{k,j}$;

(iii) f has an attracting fixed point at -2.

An entire function satisfying (i) and (iii) exists by Lemma 3.3.3, and (ii) can be deduced from (i).

Eremenko and Lyubich then show that the set

$$K := \left\{ z \in \mathbb{C} : \; f^k(z) \in \bigcup_{j=1}^{4} \overline{Q_{k,j}} \text{ for all } k \geq 0 \right\}$$

is a subset of $\mathcal{J}(f)$ that has positive measure. See Figure 3.3 for an illustration of this approach, stated in terms of the functions $\Phi_{k,j}$ approximated by f.

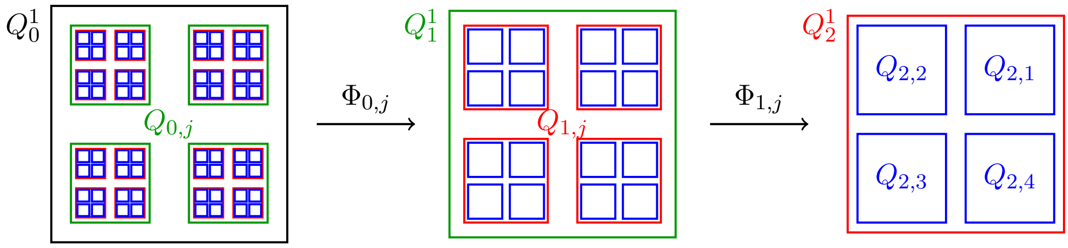


Figure 3.3: An illustration of the mapping behaviour of the maps $\Phi_{k,j}$ and sets of the form $\{z \in Q_{0,j_0} : \Phi_{k,j_k} \circ \Phi_{k-1,j_{k-1}} \circ ... \circ \Phi_{0,j_0}(z) \in \bigcup_{j=1}^{4} Q_{k+1,j}\}$.

We now explain how to extend the above construction to ensure that $\mathcal{J}(f)$ is thin at infinity. Note that $Q_{k,j} \subset \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$ for all $k \geq 0$ and all $j \in \{1, ..., 4\}$. For $k \in \mathbb{Z}$ and $l \in \mathbb{Z} \setminus \{0\}$, let $R_{k,l}$ be the square defined as

$$R_{k,l} := \{z \in \mathbb{C} : |\operatorname{Re} z - 4k| < 1, |\operatorname{Im} z - 4l| < 1\}.$$

In fact, Lemma 3.3.3 yields that, in addition to (i)-(iii), f can be chosen to satisfy

$$|f(z) + 2| < \frac{1}{2} \text{ for } z \in \mathcal{D}\left(-2, \frac{1}{2}\right) \cup \bigcup_{k,l \in \mathbb{Z}\setminus\{0\}} R_{k,l}.$$

Then $f(\mathcal{D}(-2, 1/2)) \subset \mathcal{D}(-2, 1/2)$, so that $\mathcal{D}(-2, 1/2) \subset \mathcal{F}(f)$ by Montel's theorem (Theorem 2.4.2). Since $f(R_{k.l}) \subset \mathcal{D}(-2, 1/2)$, we deduce that $R_{k,l} \subset \mathcal{F}(f)$ for all $k \in \mathbb{Z}$ and $l \in \mathbb{Z} \setminus \{0\}$. Thus $\mathcal{J}(f)$ is thin at infinity.

Chapter 4

A class of Newton maps with Julia sets of Lebesgue measure zero

In this chapter we prove Theorem B. First, in Section 4.1, we recall the statements of Theorem B and Corollary C, and discuss a result of Bergweiler that together with Theorem B implies Corollary C. In Sections 4.2-4.10, we prove Theorem B. We begin by introducing the change of variables $w = q(z)$ in Section 4.2. Then, in Section 4.3, we give asymptotic representations of g and f. In Sections 4.4-4.9 we mainly study the function obtained from f by the change of variables $w = q(z)$, using ideas of Jankowski [Jan96, Chapter 3]. The first of these sections, Section 4.4, introduces a class of subsets of $\mathbb{C}$ related to the mapping behaviour of the function under consideration, Section 4.5 concerns the postsingular set of f, and in Sections 4.6-4.9, we investigate the location and size of $q(\mathcal{F}(f))$. We complete the proof of Theorem B in Section 4.10. Finally, in Section 4.11, we discuss examples of functions satisfying the assumptions of Theorem B or Corollary C.

4.1 Convergence of Newton's method in a dense subset of the plane

In this section we discuss some results of Bergweiler related to Theorem B and Corollary C. Recall that we consider functions of the form

$$g(z) = \int_0^z p(t)e^{q(t)}dt + c, \tag{4.1.1}$$

where q is a non-constant polynomial, p is a polynomial with $p \not\equiv 0$, and $c \in \mathbb{C}$. Recall that the main result of this chapter, Theorem B, says the following.

Theorem B. *Let g be of the form* (4.1.1) *but not of the form $\tilde{p}(z)e^{\tilde{q}(z)}$ with polynomials $\tilde{p}$ and $\tilde{q}$, and let f be the function from Newton's method for g. Denote the zeros of g'' which are not zeros of g or g' by $z_1, ..., z_N$. Suppose that for each $j \in \{1, ..., N\}$, either the point z_j is attracted by a periodic cycle of f, or there exists $n \in \mathbb{N}$ with $f^n(z_j) = \infty$. Then the Lebesgue measure of $\mathcal{J}(f)$ is zero.*

Bergweiler proved the following result [Ber93b, Theorem 3].

Theorem 4.1.1. *Let g be of the form* (4.1.1) *but not of the form e^{az+b} with $a, b \in \mathbb{C}$, and let f be the function from Newton's method for g. Denote the zeros of g'' which are not* *zeros of g or g' by $z_1, ..., z_N$. If $f^n(z_j)$ converges to a finite limit for all $j \in \{1, ..., N\}$, then $f^n(z)$ converges to zeros of g in the entire Fatou set of f, which is an open dense subset of the complex plane.*

If $f^n(z_j)$ converges to a finite limit, then this limit is a fixed point of f and hence a zero of g. So the theorem says that $f^n(z)$ converges to zeros of g in an open dense subset of the complex plane if this is the case for each z_j. Theorem 4.1.1 together with Theorem B immediately implies Corollary C, whose statement we recall below.

Corollary C. *Let g be of the form* (4.1.1) *but not of the form $\tilde{p}(z)e^{\tilde{q}(z)}$ with polynomials $\tilde{p}$ and $\tilde{q}$, and let f be the function from Newton's method for g. Denote the zeros of g'' which are not zeros of g or g' by $z_1, ..., z_N$. If $f^n(z_j)$ converges to a finite limit for all $j \in \{1, ..., N\}$, then $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$.*

Bergweiler's proof of Theorem 4.1.1 is based on the two following theorems, which are special cases of [Ber93b, Theorems 1 and 2].

Theorem 4.1.2. *Let g be of the form* (4.1.1)*, and let f be the function from Newton's method for g. Then f does not have wandering domains.*

Theorem 4.1.3. *Let g be of the form* (4.1.1)*, and let f be the function from Newton's method for g. Then every cycle of Baker domains of f contains a singular value of f.*

Sullivan [Sul85] showed that rational functions do not have wandering domains. Transcendental meromorphic functions in general may have wandering domains; see, for example, [Bak76, Bak84, Bak85, EL87, BKL90].

In Bergweiler's proofs of Theorems 4.1.2 and 4.1.3, simply connected wandering domains and cycles of simply connected Baker domains not containing any singular value of f are ruled out using quasiconformal deformation techniques from Sullivan's proof that rational functions do not have wandering domains [Sul85]. The proofs in the multiply-connected case also rely on the absence of singular values in the respective domains and specific properties of f. More recently, it has been established in a series of papers that for a certain class of meromorphic functions, including all Newton maps of entire functions, all Fatou components are simply connected [BT96, FJT08, FJT11, BFJK14]. See also [BFJK18] for a unified approach in the case of Newton's method for entire functions. However, Theorems 1 and 2 in [Ber93b] are also stated for a more general class of functions, which includes functions not covered by the results in [BT96, FJT08, FJT11, BFJK14].

Before we turn to the proof of our Theorem B, we outline the idea of Bergweiler's proof of Theorem 4.1.1.

Sketch of proof of Theorem 4.1.1. First, Bergweiler proves that f does not have finite asymptotic values. See also Lemma 4.5.1 for a more elementary proof of this fact. Thus every singular value of f is a critical value of f; that is, a zero of g or g'' but not a zero of g'. Under the assumptions of Theorem 4.1.1, this yields that all singular values of f are contained in attractive basins of fixed points of f. In particular, by Theorem 4.1.3, f does not have Baker domains. By Lemma 2.4.22, each cycle of attractive basins or parabolic domains contains a singular value. So f has neither parabolic domains nor cycles of attractive basins of period at least two. Moreover, the boundary of Siegel disks and Herman rings is always contained in the postsingular set, which is impossible under the assumptions of Theorem 4.1.1. Wandering domains are ruled out by Theorem 4.1.2. By the classification of Fatou components (Lemma 2.4.17), each Fatou component of f must be contained in an attractive basin of a fixed point of f. But the fixed points of f are precisely the zeros of g, so $f^n(z)$ converges to zeros of g in the entire Fatou set of f. □

4.2 A change of variables

This section introduces a change of variables that will be used in the proof of Theorem B. Throughout Sections 4.2-4.10 let g be defined by (4.1.1), and let f be the function from Newton's method for g. The following remark helps to simplify notation.

Remark 4.2.1. Suppose that $q(t) = at^d + O(t^{d-1})$ as $t \to \infty$ with $a \in \mathbb{C} \setminus \{0\}$ and $d \geq 1$. Let $\alpha \in \mathbb{C}$ with $\alpha^d = a$. Then $q(t/\alpha) = t^d + O(t^{d-1})$ as $t \to \infty$,

$$g(z/\alpha) = \int_0^{z/\alpha} p(t)e^{q(t)}\,dt + c = \int_0^z \frac{1}{\alpha}p\left(\frac{t}{\alpha}\right)e^{q(t/\alpha)}\,dt + c,$$

and the function $z \mapsto \alpha z$ conjugates the Newton map of $g(z/\alpha)$ to the Newton map of g. Thus we can and will assume without loss of generality that $a = 1$; that is,

$$q(t) = t^d + O(t^{d-1})$$

as $t \to \infty$.

Also, since the functions g and $b \cdot g$ for $b \in \mathbb{C} \setminus \{0\}$ have the same zeros and the same Newton map, we can and will assume without loss of generality that p has the form

$$p(t) = dt^m + O(t^{m-1})$$

as $t \to \infty$, where $d = \deg(q)$.

Let $R > 0$ such that all finite critical values of q are contained in $\mathcal{D}(0, R)$, and such that for $|z| \geq (1/2)R^{1/d}$, we have

$$\frac{1}{2^d}|z|^d \leq |q(z)| \leq 2^d|z|^d. \tag{4.2.1}$$

Set

$$\mathcal{G} := \mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup [0, \infty)).$$

Lemma 4.2.2. *There exists $a > 0$ such that the set $q^{-1}(\mathcal{G})$ consists of d components $\mathcal{S}_1, ..., \mathcal{S}_d$ satisfying*

$$\mathcal{S}_j \subset \left\{z \in \mathbb{C} : |z| > \frac{1}{2}R^{1/d}, \ \frac{2(j-1)\pi}{d} - \frac{a}{|z|} < \arg(z) < \frac{2j\pi}{d} + \frac{a}{|z|}\right\}$$

and

$$\mathcal{S}_j \supset \left\{z \in \mathbb{C} : |z| > 2R^{1/d}, \ \frac{2(j-1)\pi}{d} + \frac{a}{|z|} < \arg(z) < \frac{2j\pi}{d} - \frac{a}{|z|}\right\}$$

for all $j \in \{1, ..., d\}$. Moreover, q maps each $\mathcal{S}_j$ conformally onto $\mathcal{G}$.

Proof. Since $\mathcal{G}$ is simply connected and contains no critical values of q, its preimage $q^{-1}(\mathcal{G})$ consists of d components, and q maps each of them conformally onto $\mathcal{G}$. By (4.2.1),

$$q\left(\mathcal{D}\left(0, \frac{1}{2}R^{1/d}\right)\right) \subset \mathcal{D}(0, R)$$

and

$$q(\mathbb{C} \setminus \mathcal{D}(0, 2R^{1/d})) \subset \mathbb{C} \setminus \mathcal{D}(0, R).$$

Also, for $z \in \mathbb{C}$, we have

$$\arg(q(z)) = \arg\left(z^d\left(1 + O\left(\frac{1}{z}\right)\right)\right) \equiv d\arg(z) + \arg\left(1 + O\left(\frac{1}{z}\right)\right) \\ \equiv d\arg(z) + O\left(\frac{1}{z}\right) \mod 2\pi$$

as $z \to \infty$. Thus

$$\arg(z) \equiv \frac{\arg(q(z))}{d} + O\left(\frac{1}{z}\right) \mod \frac{2\pi}{d}$$

as $z \to \infty$. Because q is surjective, this implies the desired conclusion. $\square$

For $j \in \{1, ..., d\}$, let φ_j denote the branch of q^{-1} defined in $\mathcal{G}$ with $\varphi_j(\mathcal{G}) = \mathcal{S}_j$.

4.3 The asymptotics of g and f

This section provides asymptotic representations for $g(\varphi_j(w)), g(z), f(\varphi_j(w))$ and $f(z)$. Let

$$\lambda := \frac{d-1-m}{d}.$$

Then

$$\frac{p(z)}{q'(z)} = z^{-\lambda d}\left(1 + O\left(\frac{1}{z}\right)\right) \tag{4.3.1}$$

as $z \to \infty$ and, for $j \in \{1, ..., d\}$,

$$\left|\frac{p(\varphi_j(w))}{q'(\varphi_j(w))}\right| = |w|^{-\lambda}\left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right) \tag{4.3.2}$$

as $w \to \infty$ in $\mathcal{G}$.

Lemma 4.3.1. *Let $j \in \{1, ..., d\}$. Then there exists $c_j \in \mathbb{C}$ such that*

$$g(\varphi_j(w)) = c_j + \frac{p(\varphi_j(w))}{q'(\varphi_j(w))}\left(1 + \frac{\lambda}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right)\right)e^w$$

as $w \to \infty$ in $\mathcal{G}$.

In terms of $z = \varphi_j(w)$, Lemma 4.3.1 says the following.

Corollary 4.3.2. *For $j \in \{1, ..., d\}$, we have*

$$g(z) = c_j + \frac{p(z)}{q'(z)}\left(1 + \frac{\lambda}{z^d} + O\left(\frac{1}{z^{d+1}}\right)\right)e^{q(z)}$$

as $z \to \infty$ in $\mathcal{S}_j$.

Remark 4.3.3. The numbers c_j, for $j \in \{1, ..., d\}$, are precisely the finite asymptotic values of g.

Proof of Lemma 4.3.1. Let $x_0 \in (-\infty, -R) = \mathcal{G} \cap (-\infty, 0]$ and $w \in \mathcal{G}$. Then

$$\begin{aligned} g(\varphi_j(w)) &= \int_0^{\varphi_j(w)} p(t)e^{q(t)}\,dt + c \\ &= \int_{\varphi_j(x_0)}^{\varphi_j(w)} p(t)e^{q(t)}\,dt + \int_0^{\varphi_j(x_0)} p(t)e^{q(t)}\,dt + c \\ &= \int_{\varphi_j(x_0)}^{\varphi_j(w)} p(t)e^{q(t)}\,dt + g(\varphi_j(x_0)) \\ &= \int_{x_0}^{w} \varphi_j'(s)p(\varphi_j(s))e^s\,ds + g(\varphi_j(x_0)). \end{aligned}$$

Set

$$r(s) := \varphi_j'(s)p(\varphi_j(s)) = \frac{p(\varphi_j(s))}{q'(\varphi_j(s))}.$$

Repeated integration by parts yields

$$\int_{x_0}^{w} r(s)e^s\,ds = (r(s) - r'(s) + r''(s))e^s\Big|_{x_0}^{w} - \int_{x_0}^{w} r'''(s)e^s\,ds.$$

We have

$$\begin{aligned}
r'(s) &= \varphi_j'(s)\frac{q'(\varphi_j(s))p'(\varphi_j(s)) - q''(\varphi_j(s))p(\varphi_j(s))}{q'(\varphi_j(s))^2}\\
&= \left(\frac{1}{q'(\varphi_j(s))}\cdot\frac{p(\varphi_j(s))}{q(\varphi_j(s))}\right)\cdot\left(\frac{q(\varphi_j(s))}{p(\varphi_j(s))}\cdot\frac{q'(\varphi_j(s))p'(\varphi_j(s)) - q''(\varphi_j(s))p(\varphi_j(s))}{q'(\varphi_j(s))^2}\right)\\
&= \frac{p(\varphi_j(s))}{q'(\varphi_j(s))s}\cdot\frac{q(\varphi_j(s))q'(\varphi_j(s))p'(\varphi_j(s))/p(\varphi_j(s)) - q(\varphi_j(s))q''(\varphi_j(s))}{q'(\varphi_j(s))^2}\\
&= \frac{r(s)}{s}\cdot\frac{m-(d-1)}{d}\left(1 + O\left(\frac{1}{|s|^{1/d}}\right)\right)\\
&= -\frac{\lambda}{s}r(s)\left(1 + O\left(\frac{1}{|s|^{1/d}}\right)\right).
\end{aligned}$$

Also, a computation shows that

$$r''(s) = r(s)O\left(\frac{1}{s^2}\right) \quad \text{and } r'''(s) = r(s)O\left(\frac{1}{s^3}\right)$$

as $|s| \to \infty$. With $h(x_0) := (r(x_0) - r'(x_0) + r''(x_0))e^{x_0}$, we obtain

$$\int_{x_0}^{w} r(s)e^s\,ds = r(w)e^w\left(1 + \frac{\lambda}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right)\right) - h(x_0) - \int_{x_0}^{w} r'''(s)e^s\,ds.$$

We have

$$\int_{x_0}^{w} r'''(s)e^s\,ds = \int_{-|w|}^{w} r'''(s)e^s\,ds + \int_{-\infty}^{-|w|} r'''(s)e^s\,ds - \int_{-\infty}^{x_0} r'''(s)e^s\,ds.$$

To estimate $\int_{-|w|}^{w} r'''(s)e^s\,ds$, let γ be the part of the circle centred at zero with radius $|w|$ connecting $-|w|$ and w in $\mathcal{G}$. Then $\operatorname{Re} s \le \operatorname{Re} w$ for $s \in \gamma$. We deduce that

$$\begin{aligned}
\left|\int_{-|w|}^{w} r'''(s)e^s\,ds\right| &\le \operatorname{length}(\gamma)\cdot\max_{s\in\gamma}|r'''(s)e^s| \le O(|w|)|r(w)|O\left(\frac{1}{|w|^3}\right)e^{\operatorname{Re} w}\\
&= |r(w)|O\left(\frac{1}{|w|^2}\right)|e^w|.
\end{aligned}$$

Let us now estimate $\int_{-\infty}^{-|w|} r'''(s)e^s\,ds$. By (4.3.2), we have $|r(s)| = |s|^{-\lambda}(1 + o(1))$ as $|s| \to \infty$. First suppose that $\lambda \ge 0$. Using that $r'''(s) = r(s)O(1/s^3)$, we deduce that

$$\begin{aligned}
\left|\int_{-\infty}^{-|w|} r'''(s)e^s\,ds\right| &\le |r(w)|e^{-|w|}O\left(\frac{1}{|w|^3}\right)\int_{-\infty}^{-|w|} e^{s+|w|}\,ds\\
&\le |r(w)e^w|O\left(\frac{1}{|w|^3}\right)\int_{-\infty}^{0} e^s\,ds = |r(w)e^w|O\left(\frac{1}{|w|^3}\right).
\end{aligned}$$

Now suppose that $\lambda < 0$. Then

$$\left| \int_{-\infty}^{-|w|} r'''(s) e^s \, ds \right| \leq O\left(\frac{1}{|w|^3}\right) \int_{-\infty}^{-|w|} |s|^{-\lambda} e^s \, ds.$$

Integration by parts yields

$$\int_{-\infty}^{-|w|} |s|^{-\lambda} e^s \, ds = O(|w|^{-\lambda} e^{-|w|}) \leq O(|r(w) e^w|)$$

and hence

$$\int_{-\infty}^{-|w|} r'''(s) e^s \, ds = r(w) e^w O\left(\frac{1}{|w|^3}\right).$$

Altogether, we obtain the desired conclusion with

$$c_j = g(\varphi_j(x_0)) - h(x_0) + \int_{-\infty}^{x_0} r'''(s) e^s \, ds. \qquad \square$$

For the Newton map f of g, Lemma 4.3.1 implies the following.

Corollary 4.3.4. *For $j \in \{1, ..., d\}$, we have*

$$f(\varphi_j(w)) = \varphi_j(w) - \frac{1}{q'(\varphi_j(w))} \left(1 + \frac{\lambda}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right)\right) - \frac{c_j e^{-w}}{p(\varphi_j(w))}$$

as $w \to \infty$ in $\mathcal{G}$.

In terms of z, Corollary 4.3.4 says the following.

Corollary 4.3.5. *For $j \in \{1, ..., d\}$, we have*

$$f(z) = z - \frac{1}{q'(z)} \left(1 + \frac{\lambda}{z^d} + O\left(\frac{1}{z^{d+1}}\right)\right) - \frac{c_j e^{-q(z)}}{p(z)}$$

as $z \to \infty$ in $\mathcal{S}_j$.

4.4 Partitioning the plane

For a more detailed study of the behaviour of $f \circ \varphi_j$, we divide the complex plane in several sets, depending on how large $|e^{-w}|$ is compared to some power of $|w|$. More precisely, we consider sets whose boundary points satisfy

$$\operatorname{Re} w = \mu \log |w| - \log \alpha \tag{4.4.1}$$

for certain $\mu \in \mathbb{R}$ and $\alpha > 0$. In this section we mainly discuss properties of curves satisfying (4.4.1). Such curves, and sets bounded by those curves, were introduced by Jankowski [Jan96, §3.3.4]. Although our results are similar to those in [Jan96], we provide their proofs for completeness. While Jankowski parametrised the curves defined by (4.4.1) by $x = \operatorname{Re} z$, we will use a parametrisation by $y = \operatorname{Im} z$, leading to differences in the proofs.

The next lemma yields that if $|y|$ is sufficiently large, there exists a unique $x_y \in \mathbb{R}$ such that $w = x_y + iy$ satisfies (4.4.1).

Lemma 4.4.1. *Let $\mu \in \mathbb{R}$, $\alpha > 0$ and $y \in \mathbb{R}$ with $|y| \geq 2|\mu|$. Then there exists a unique $x_y \in \mathbb{R}$ with*

$$x_y = \mu \log |x_y + iy| - \log \alpha. \tag{4.4.2}$$

If $x > x_y$, then

$$x > \mu \log |x + iy| - \log \alpha. \tag{4.4.3}$$

If $x < x_y$, then

$$x < \mu \log |x + iy| - \log \alpha. \tag{4.4.4}$$

Proof. Let $\varphi_y : \mathbb{R} \to \mathbb{R}$,

$$\varphi_y(x) = x - \mu \log |x + iy| = x - \frac{\mu}{2} \log(x^2 + y^2).$$

Then $\varphi_y(x) \to \infty$ as $x \to \infty$, and $\varphi_y(x) \to -\infty$ as $x \to -\infty$. Thus φ_y is surjective, so there exists x_y satisfying (4.4.2).

Also,

$$\varphi_y'(x) = 1 - \frac{\mu x}{x^2 + y^2}.$$

Since

$$\frac{|\mu x|}{x^2 + y^2} \leq |\mu| \frac{\max\{|x|, |y|\}}{\max\{x^2, y^2\}} = \frac{|\mu|}{\max\{|x|, |y|\}} \leq \frac{1}{2},$$

we have

$$\varphi_y'(x) \geq \frac{1}{2}.$$

Thus φ_y is strictly increasing, which implies (4.4.3), (4.4.4) and uniqueness of x_y. □

For $\mu \in \mathbb{R}$ and $\alpha > 0$, let

$$\gamma_{\mu,\alpha} : (-\infty, -2|\mu|] \cup [2|\mu|, \infty) \to \mathbb{R}, \; \gamma_{\mu,\alpha}(y) = x_y.$$

See Figure 4.1 for an illustration of such curves. The next lemma states some properties of the curves $\gamma_{\mu,\alpha}$.

Lemma 4.4.2. *Let $\mu \in \mathbb{R}$ and $\alpha > 0$.*

(i) The function $\gamma_{\mu,\alpha}$ is continuously differentiable.

(ii) If $\mu > 0$, then $\lim_{|y|\to\infty} \gamma_{\mu,\alpha}(y) = \infty$. If $\mu < 0$, then $\lim_{|y|\to\infty} \gamma_{\mu,\alpha}(y) = -\infty$. For $\mu = 0$, we have $\gamma_{\mu,\alpha} \equiv -\log \alpha$.

(iii) $|\gamma_{\mu,\alpha}'(y)| \leq 2|\mu|/|y|$. In particular, $\lim_{|y|\to\infty} \gamma_{\mu,\alpha}'(y) = 0$ uniformly in α.

(iv) For $\alpha > \beta > 0$, we have

$$\frac{2}{3} \log \frac{\alpha}{\beta} \leq \gamma_{\mu,\beta}(y) - \gamma_{\mu,\alpha}(y) \leq 2 \log \frac{\alpha}{\beta}$$

and

$$\lim_{|y|\to\infty} (\gamma_{\mu,\beta}(y) - \gamma_{\mu,\alpha}(y)) = \log \frac{\alpha}{\beta}.$$

Remark 4.4.3. Similar results were obtained by Jankowski [Jan96]. Our estimate for the derivative is more precise than the one in [Jan96]. Corresponding to (iv), Jankowski shows that, for $\varepsilon > 0$,

$$(1 - \varepsilon) \log \frac{\alpha}{\beta} - \varepsilon \leq \gamma_{\mu,\beta}(y) - \gamma_{\mu,\alpha}(y) \leq (1 + \varepsilon) \log \frac{\alpha}{\beta} + \varepsilon$$

if α and β are either both sufficiently large or both sufficiently small.

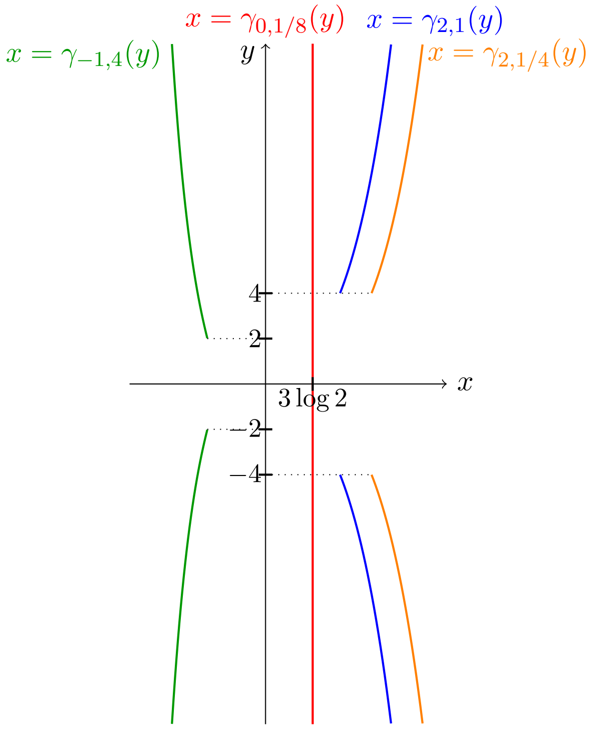


Figure 4.1: An illustration of the curves $\gamma_{\mu,\alpha}$.

Proof of Lemma 4.4.2. For $\mu = 0$, the results are obvious. We provide the proof for $\mu > 0$, the case $\mu < 0$ is analogous. To prove (i)-(iii), note that the condition

$$x = \mu \log |x + iy| - \log \alpha$$

is equivalent to

$$y^2 = \alpha^{2/\mu} e^{(2/\mu)x} - x^2.$$

The function

$$\psi(x) = \alpha^{2/\mu} e^{(2/\mu)x} - x^2$$

satisfies

$$\lim_{x\to-\infty} \psi(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} \psi(x) = \infty. \tag{4.4.5}$$

Let $x_0 := \max\{x \in \mathbb{R} : \psi(x) = 4\mu^2\}$. Then $\psi(x) > 4\mu^2$ for $x > x_0$. Also,

$$\psi'(x) = \frac{2}{\mu}\alpha^{2/\mu} e^{(2/\mu)x} - 2x = \frac{2}{\mu}(\psi(x) + x^2 - \mu x).$$

It is not difficult to see that $x^2 - \mu x \geq -\mu^2/4$ for all $x \in \mathbb{R}$. Thus

$$\psi'(x) \geq \frac{2}{\mu}\left(\psi(x) - \frac{\mu^2}{4}\right) > 0 \tag{4.4.6}$$

for $x > x_0$. In particular, $\psi : [x_0, \infty) \to [4\mu^2, \infty)$ is bijective. This implies that

$$\gamma_{\mu,\alpha}(y) = \psi^{-1}(y^2)$$

is a continuously differentiable function. By (4.4.5), property (ii) is satisfied. Also, using (4.4.6) and the fact that $y^2 \geq 4\mu^2$, we deduce

$$|\gamma'_{\mu,\alpha}(y)| = \left|\frac{2y}{\psi'(\psi^{-1}(y^2))}\right| \leq \frac{2|y|}{(2/\mu)(\psi(\psi^{-1}(y^2)) - \mu^2/4)} = \frac{\mu|y|}{y^2 - \mu^2/4} \leq \frac{2\mu}{|y|}.$$

So (iii) is satisfied. To prove (iv), fix $y \in \mathbb{R}$ with $|y| \geq 2\mu$, and let φ_y be as in the proof of Lemma 4.4.1. Let $x_1 := \gamma_{\mu,\alpha}(y)$ and $x_2 := \gamma_{\mu,\beta}(y)$. By the mean value theorem, there exists $\xi \in [x_1, x_2]$ with

$$\log \frac{\alpha}{\beta} = \varphi_y(x_2) - \varphi_y(x_1) = \varphi_y'(\xi)(x_2 - x_1).$$

In the proof of Lemma 4.4.1, we have seen that $\varphi_y'(\xi) \geq 1/2$, and the same argument shows that $\varphi_y'(\xi) \leq 3/2$. Also, $\varphi_y'(\xi) \to 1$ as $|y| \to \infty$. □

For $\mu \in \mathbb{R}$, $\alpha > 0$ and $\nu \geq 2|\mu|$, define

$$\begin{aligned}\mathcal{H}(\mu, \alpha, \nu) &:= \{w \in \mathbb{C} : \operatorname{Re} w \geq \mu \log |w| - \log \alpha,\ |\operatorname{Im} w| \geq \nu\} \\ &= \{x + iy :\ |y| \geq \nu,\ x \geq \gamma_{\mu,\alpha}(y)\}.\end{aligned}$$

Also, set

$$\begin{aligned}\Gamma(\mu, \alpha) &:= \{w \in \mathbb{C} : \operatorname{Re} w = \mu \log |w| - \log \alpha,\ |\operatorname{Im} w| \geq 2|\mu|\} \\ &= \{\gamma_{\mu,\alpha}(y) + iy :\ |y| \geq 2|\mu|\}.\end{aligned}$$

Remark 4.4.4. For $w \in \Gamma(\mu, \alpha)$, we have

$$|e^{-w}| = e^{-\operatorname{Re} w} = \alpha |w|^{-\mu};$$

for $w \in \mathcal{H}(\mu, \alpha, \nu)$, we have

$$|e^{-w}| \leq \alpha |w|^{-\mu};$$

and for $w \in \mathbb{C} \setminus \mathcal{H}(\mu, \alpha, \nu)$ with $|\operatorname{Im} w| \geq \nu$, we have

$$|e^{-w}| > \alpha |w|^{-\mu}.$$

Recall that we will consider the change of variables $w = q(z)$ (see Section 4.2). For later reference, we state certain properties of points z for which $|e^{q(z)}|$ is bounded by some power of $|z|$.

Lemma 4.4.5. *Let U be an unbounded subset of $\mathbb{C}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that for all $z \in U$ with $|z| \geq c$, we have $|z|^a \leq |e^{q(z)}| \leq |z|^b$. Then*

$$q(z) = i \operatorname{Im} q(z)(1 + o(1))$$

and

$$\arg(z) \equiv \operatorname{sgn}(\operatorname{Im} q(z)) \frac{\pi}{2d} + o(1) \mod \frac{2\pi}{d}$$

as $z \to \infty$ in U. In particular, if $U \subset \mathcal{S}_j$ for some $j \in \{1, ..., d\}$, then

$$\arg(z) = \begin{cases} \pi/(2d) + 2\pi(j-1)/d + o(1) & \text{if } \operatorname{Im} q(z) > 0 \\ -\pi/(2d) + 2\pi j/d + o(1) & \text{if } \operatorname{Im} q(z) < 0 \end{cases} \tag{4.4.7}$$

as $z \to \infty$ in U.

Remark 4.4.6. Suppose V is an unbounded subset of $\Gamma(\mu, \alpha)$ or, more generally, $\mathcal{H}(\mu, \alpha, \nu) \setminus \mathcal{H}(\mu', \alpha', \nu)$, for some $\mu, \mu', \alpha, \alpha', \nu > 0$. Then Remark 4.4.4 yields that $U = \varphi_j(V)$ satisfies the assumptions of Lemma 4.4.5, and hence (4.4.7) holds for $z = \varphi_j(w)$.

Proof of Lemma 4.4.5. We have

$$|\operatorname{Re} q(z)| = |\log |e^{q(z)}|| \leq \max\{|a|, |b|\} \log |z| = o(|q(z)|)$$

as $z \to \infty$ in U. This implies that $q(z) = i \operatorname{Im} q(z)(1 + o(1))$ and

$$\arg(q(z)) \equiv \operatorname{sgn}(\operatorname{Im} q(z))\frac{\pi}{2} + o(1) \mod 2\pi.$$

Since

$$\arg(q(z)) = \arg(z^d(1 + o(1))) \equiv d \arg(z) + o(1) \mod 2\pi,$$

we deduce that

$$\arg(z) \equiv \operatorname{sgn}(\operatorname{Im} q(z))\frac{\pi}{2d} + o(1) \mod \frac{2\pi}{d}$$

as $z \to \infty$ in U. □

4.5 The singular values of f

Recall that

$$g(z) = \int_0^z p(t)e^{q(t)}\, dt + c,$$

where $q(t) = t^d + O(t^{d-1})$ and $p(t) = dt^m + O(t^{m-1})$ as $t \to \infty$; and f is the function from Newton's method for g.

This section concerns the singular values of f.

Lemma 4.5.1. *The function f is unbounded on any curve tending to infinity. In particular, f does not have finite asymptotic values.*

The fact that f does not have finite asymptotic values was proved by Bergweiler in [Ber93b, p. 238]. We give a more elementary proof of this result.

Proof. Suppose there exists a curve $\sigma : [0, \infty) \to \mathbb{C}$ such that $\sigma(t) \to \infty$ as $t \to \infty$ and $f(z) = O(1)$ as $z \to \infty$ in σ. Let $j \in \{1, ..., d\}$ such that $\sigma \cap \mathcal{S}_j$ is unbounded. By Corollary 4.3.5 and the assumption, we have

$$f(z) = z + O(1) - \frac{c_j e^{-q(z)}}{p(z)} = O(1)$$

as $z \to \infty$ in $\mathcal{S}_j \cap \sigma$. This implies that $c_j \neq 0$ and

$$e^{q(z)} = \frac{c_j}{p(z)(z + O(1))} = \frac{c_j}{d} z^{-(m+1)}(1 + o(1)) \tag{4.5.1}$$

as $z \to \infty$ in $\sigma \cap \mathcal{S}_j$. By Lemma 4.4.5, this yields

$$\arg(z) \equiv \operatorname{sgn}(\operatorname{Im} q(z))\frac{\pi}{2d} + o(1) \mod \frac{2\pi}{d} \tag{4.5.2}$$

as $z \to \infty$ in $\sigma \cap \mathcal{S}_j$. Because σ is continuous, we deduce that $\sigma(t) \in \mathcal{S}_j$ for all large t. Moreover, by (4.5.1) and (4.5.2),

$$\operatorname{Im} q(z) \equiv \arg\left(\frac{c_j}{d}\right) - (m+1)\arg(z) + o(1) \equiv \arg\left(\frac{c_j}{d}\right) - (m+1)\operatorname{sgn}(\operatorname{Im} q(z))\frac{\pi}{2d} + o(1)$$

mod $2\pi/d$ as $z \to \infty$ in σ. On the other hand, Lemma 4.4.5 yields that $|\operatorname{Im} q(z)| \to \infty$ as $z \to \infty$ in σ, so that $|\operatorname{Im} q(z)|$ takes any sufficiently large value. This is a contradiction. □

By Lemma 4.5.1, each singular value of f in $\mathbb{C}$ lies in the closure of the set of critical values of f. The set of critical points of f consists of

1. the zeros of g that are not zeros of g'. These are superattracting fixed points of f and form a discrete subset of $\mathbb{C}$;
2. the zeros of g'' that are not zeros of g or g'. There are only finitely many of these, $z_1, ..., z_N$, and in Theorem B we assume that each z_j is either attracted by a periodic cycle or eventually mapped to infinity.

In particular, the set of critical values of f does not have finite limit points. So every singular value of f in $\mathbb{C}$ is a critical value, and all but finitely many of them are superattracting fixed points.

Lemma 4.5.2. *Under the assumptions of Theorem B, the set $\mathcal{P}(f) \cap \mathcal{J}(f)$ is finite.*

Proof. Since the superattracting fixed points of f lie in the Fatou set and form a discrete subset of the plane, $\mathcal{P}(f) \cap \mathcal{J}(f)$ is contained in the closure of $\mathcal{O}^+(\{z_1, ..., z_N\})$. Each z_j is either attracted by a periodic cycle or eventually mapped to infinity. In particular, $\mathcal{O}^+(z_j) \cap \mathbb{C}$ is bounded and $\overline{\mathcal{O}^+(z_j)} \setminus \mathcal{O}^+(z_j)$ is finite. If $z_j \in \mathcal{J}(f)$ and z_j is attracted by a periodic cycle C, then by Lemma 2.4.14, z_j is eventually mapped to C, so the forward orbit of z_j is finite. □

Recall that

$$g(z) = c_j + \frac{p(z)}{q'(z)} \left(1 + \frac{\lambda}{z^d} + O\left(\frac{1}{z^{d+1}}\right)\right) e^{q(z)}$$

as $z \to \infty$ in $\mathcal{S}_j$, for $j \in \{1, ..., d\}$. We will see later that if $c_j \neq 0$, then g has infinitely many zeros in $\mathcal{S}_j$. It is easy to see that this cannot be the case for $c_j = 0$. However, we show now that under the assumptions of Theorem B, we always have $c_j \neq 0$.

Lemma 4.5.3. *If $c_j = 0$ for some $j \in \{1, ..., d\}$, then f has a Baker domain.*

This follows from a theorem by Buff and Rückert [BR06, Theorem 4.1] which says that if an entire function has a logarithmic singularity over zero, then its Newton map has a Baker domain or a parabolic domain where the iterates tend to infinity, depending on whether the Newton map is transcendental or rational. A more elementary proof is the following.

Proof of Lemma 4.5.3. If $c_j = 0$, then Corollary 4.3.5 yields that

$$f(z) = z - \frac{1}{dz^{d-1}} + O\left(\frac{1}{z^d}\right)$$

as $z \to \infty$ in $\mathcal{S}_j$. By Lemma 2.4.19, f has a Baker domain. □

Corollary 4.5.4. *Under the assumptions of Theorem B, we have $c_j \neq 0$ for all $j \in \{1, ..., d\}$.*

Proof. By Theorem 4.1.3, every cycle of Baker domains of f contains a singular value of f. This is impossible under the assumptions of Theorem B. □

Let us assume until the end of Section 4.10, that is, until the completion of the proof of Theorem B, that g and f satisfy the assumptions of Theorem B. We now investigate the location of the zeros of g. It turns out that their images under q are close to the curves $\Gamma(\lambda, 1/|c_j|)$ defined in Section 4.4. More precisely, we have the following.

Lemma 4.5.5. *For $j \in \{1, ..., d\}$ and $k \in \mathbb{Z}$, let $v_{j,k} \in \Gamma(\lambda, 1/|c_j|)$ such that*

$$\operatorname{Im} v_{j,k} = \begin{cases} \arg(-c_j) + \lambda(\pi/2 + 2\pi(j-1)) + 2k\pi & \textit{if } k \geq 0 \\ \arg(-c_j) + \lambda(-\pi/2 + 2\pi j) + 2k\pi & \textit{if } k < 0. \end{cases}$$

If $z \in \mathcal{S}_j$ is a zero of g, then there exists $k \in \mathbb{Z}$ such that

$$q(z) = v_{j,k} + o(1) \tag{4.5.3}$$

as $z \to \infty$. Vice versa, if $j \in \{1, ..., d\}$, then g has a zero $z \in \mathcal{S}_j$ satisfying (4.5.3) as $|k| \to \infty$.

Proof. First suppose that $z \in \mathcal{S}_j$ is a zero of g. By Corollary 4.3.2 and (4.3.1),

$$g(z) = c_j + z^{-d\lambda}(1 + o(1))e^{q(z)}$$

as $z \to \infty$, and hence

$$e^{q(z)} = -c_j z^{d\lambda}(1 + o(1)). \tag{4.5.4}$$

Thus

$$\begin{aligned} \operatorname{Re} q(z) &= \log|e^{q(z)}| = \log|c_j| + d\lambda \log|z| + o(1) \\ &= \log|c_j| + \lambda \log|q(z)| + o(1) \\ &= \lambda \log|q(z)| - \log\frac{1}{|c_j|} + o(1) \end{aligned}$$

and

$$\operatorname{Im} q(z) \equiv \arg(-c_j) + d\lambda \arg(z) + o(1) \mod 2\pi. \tag{4.5.5}$$

Also, by (4.5.4), the assumptions of Lemma 4.4.5 are satisfied, so that

$$\arg(z) = \begin{cases} \pi/(2d) + 2\pi(j-1)/d + o(1) & \text{if } \operatorname{Im} q(z) > 0 \\ -\pi/(2d) + 2\pi j/d + o(1) & \text{if } \operatorname{Im} q(z) < 0. \end{cases} \tag{4.5.6}$$

Inserting (4.5.6) into (4.5.5) yields the first part of Lemma 4.5.5.

Let us now prove the second part. We provide the proof for $k > 0$, the proof for $k < 0$ is analogous. Recall that φ_j is the branch of q^{-1} that maps $\mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup [0, \infty))$ onto $\mathcal{S}_j$. Let $G_{j,k}$ be the interior of the set of all

$$v \in \mathcal{H}\left(\lambda, \frac{3}{2|c_j|}, 2|\lambda|\right) \setminus \mathcal{H}\left(\lambda, \frac{1}{2|c_j|}, 2|\lambda|\right)$$

satisfying

$$|\operatorname{Im} v - \operatorname{Im} v_{j,k}| < \pi.$$

We use the minimum principle to show that $g \circ \varphi_j$ has a zero in $G_{j,k}$. For $v = q(\varphi_j(v)) \in G_{j,k}$, Lemma 4.4.5 and the definition of $v_{j,k}$ yield

$$\arg(\varphi_j(v)) \equiv \frac{\pi}{2d} + \frac{2\pi(j-1)}{d} + o(1) \equiv \frac{\arg(-c_j) - \operatorname{Im}(v_{j,k})}{-d\lambda} + o(1) \mod \frac{2\pi}{d\lambda}. \tag{4.5.7}$$

In particular, this is true for $v = v_{j,k}$. Also, since $v_{j,k} \in \Gamma(\lambda, 1/|c_j|)$, we have

$$e^{-\operatorname{Re} v_{j,k}} = \frac{1}{|c_j|}|v_{j,k}|^{-\lambda} = \frac{1}{|c_j|}|\varphi_j(v_{j,k})|^{-d\lambda}(1 + o(1)),$$

and thus

$$|\varphi_j(v_{j,k})|^{-d\lambda} = |c_j| e^{-\operatorname{Re} v_{j,k}}(1 + o(1)).$$

Using Lemma 4.3.1 and (4.3.1), we deduce

$$\begin{aligned}(g \circ \varphi_j)(v_{j,k}) &= c_j + \varphi_j(v_{j,k})^{-d\lambda}(1 + o(1)) \exp(v_{j,k}) \\ &= c_j + |\varphi_j(v_{j,k})|^{-d\lambda} \exp(-id\lambda \arg(\varphi_j(v_{j,k})))(1 + o(1)) \exp(v_{j,k}) \\ &= c_j + |c_j| \exp(-\operatorname{Re} v_{j,k}) \exp(i(\arg(-c_j) - \operatorname{Im}(v_{j,k})))(1 + o(1)) \exp(v_{j,k}) \\ &= c_j - c_j(1 + o(1)) = o(1).\end{aligned}$$

Next, we derive a lower bound for $|g \circ \varphi_j|$ on $\partial G_{j,k}$.

If $v \in \Gamma(\lambda, 3/(2|c_j|))$, then

$$\begin{aligned}|(g \circ \varphi_j)(v)| &= |c_j + \varphi_j(v)^{-d\lambda}(1 + o(1))e^v| \geq ||c_j| - |v|^{-\lambda} e^{\operatorname{Re} v}(1 + o(1))| \\ &= \left| |c_j| - \frac{2|c_j|}{3}(1 + o(1)) \right| = \frac{|c_j|}{3} + o(1)\end{aligned}$$

as $n \to \infty$. An analogous estimate yields that if $v \in \Gamma(\lambda, 1/(2|c_j|))$, then

$$|(g \circ \varphi_j)(v)| \geq |2|c_j|(1 + o(1)) - |c_j|| = |c_j| + o(1)$$

as $n \to \infty$. If $\operatorname{Im}(v) = \operatorname{Im}(v_{j,k}) \pm \pi$, then by (4.5.7),

$$\arg(\varphi_j(v)^{-d\lambda} e^v) \equiv \arg(-c_j) \pm \pi + o(1) \equiv \arg(c_j) + o(1) \mod 2\pi.$$

Thus, for $v \in \overline{G_{j,k}}$ with $\operatorname{Im} v = \operatorname{Im} v_{j,k} \pm \pi$, we have

$$\begin{aligned}|(g \circ \varphi_j)(v)| &= |c_j + \varphi_j(v)^{-d\lambda}(1 + o(1))e^v| \\ &= ||c_j| \exp(i \arg(c_j)) + |v|^{-\lambda}|e^v| \exp(i \arg(c_j) + o(1))(1 + o(1))| \\ &= ||c_j| + |v|^{-\lambda}|e^v|(1 + o(1))| \geq |c_j|\end{aligned}$$

if k is sufficiently large. We obtain

$$|(g \circ \varphi_j)(v_{j,k})| = o(1) < \min_{v \in \partial G_{j,k}} |v|$$

if k is sufficiently large. By the minimum principle, this implies that $g \circ \varphi_j$ has a zero $w \in G_{j,k}$. The first part of the lemma yields that $z := \varphi_j(w)$ satisfies (4.5.3). □

Corollary 4.5.6. *Let $j \in \{1, ..., d\}$. Then each zero z of g in $\mathcal{S}_j$ satisfies*

$$\operatorname{dist}(z, \partial \mathcal{S}_j) \geq \left(\frac{1}{d} + o(1)\right) |z|$$

as $z \to \infty$.

Proof. By (4.5.6) and since $\sin(x) \geq (2/\pi)x$ for $x \in [0, \pi/2]$, we have

$$\operatorname{dist}(z, \partial \mathcal{S}_j) = |z| \sin\left(\frac{\pi}{2d} + o(1)\right) \geq \left(\frac{1}{d} + o(1)\right) |z|$$

as $z \to \infty$. □

4.6 The set $q(\mathcal{F}(f))$: first part

For $j \in \{1, ..., d\}$, let

$$\mathcal{F}_j := \mathcal{F}(f) \cap \mathcal{S}_j.$$

In Sections 4.6-4.8 we investigate the location and density of $q(\mathcal{F}_j)$ in three different subsets of $\mathbb{C}$, using the sets $\mathcal{H}(\mu, \alpha, \nu)$ introduced in Section 4.4. The first subset is $\mathcal{H}(\lambda, 1/|c_j|, \nu)$, the second one is $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ for small $\alpha_1 > 0$ and large $\beta_1 > 0$, and the third set is $\{w \in \mathbb{C} : |\operatorname{Im} w| \geq \nu\} \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$ for large $\beta_2 > 0$. See Figure 4.2 for an illustration of these sets.

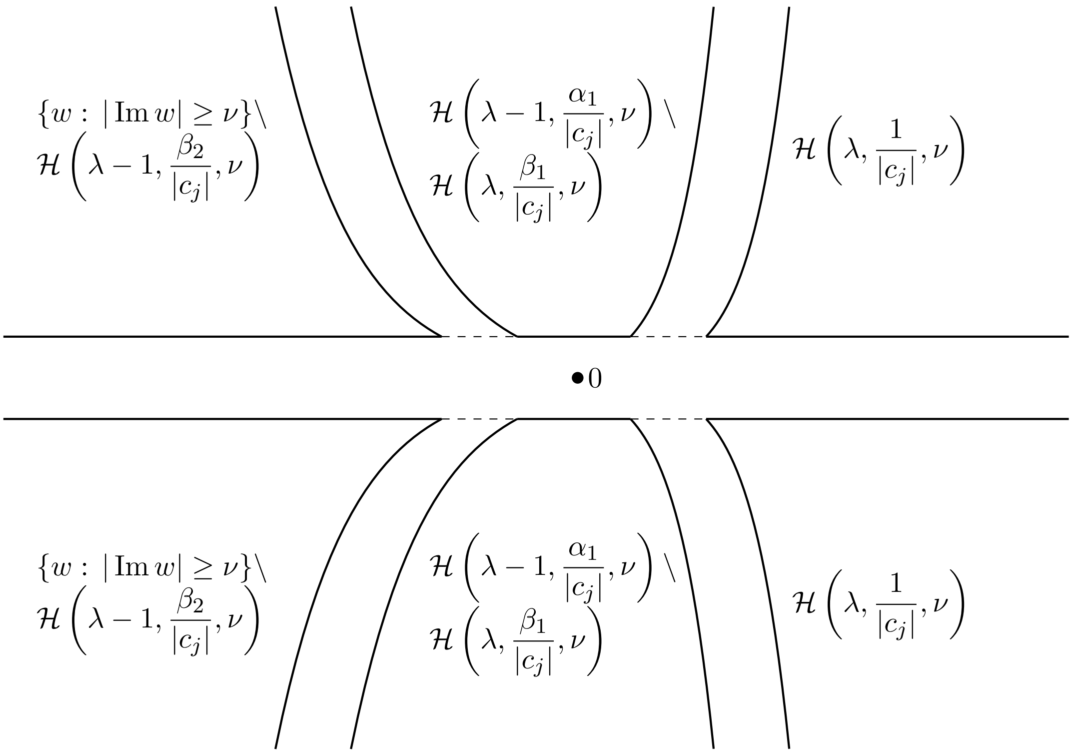


Figure 4.2: An illustration of the sets $\mathcal{H}(\lambda, 1/|c_j|, \nu)$, $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ and $\{w \in \mathbb{C} : |\operatorname{Im} w| \geq \nu\} \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$ in the case when $\lambda > 0$.

In this section we investigate the location and density of $q(\mathcal{F}_j)$ in $\mathcal{H}(\lambda, 1/|c_j|, \nu)$ for $j \in \{1, ..., d\}$ and large $\nu > 0$. Recall that the branch φ_j of q^{-1} maps $\mathcal{H}(\lambda, 1/|c_j|, \nu)$ to a subset of $\mathcal{S}_j$.

Lemma 4.6.1. *Let $j \in \{1, ..., d\}$. There exists $\nu_0 > 0$ such that*

$$(f \circ \varphi_j)(\mathcal{H}(\lambda, 1/|c_j|, \nu_0)) \subset \mathcal{S}_j.$$

In particular, if $(q \circ f \circ \varphi_j)^k(w) \in \mathcal{H}(\lambda, 1/|c_j|, \nu_0)$ for all $k \in \{0, ..., n-1\}$, then $(f^n \circ \varphi_j)(w) \in \mathcal{S}_j$ and $(q \circ f \circ \varphi_j)^n(w) = (q \circ f^n \circ \varphi_j)(w)$.

Proof. Let $w \in \mathcal{H}(\lambda, 1/|c_j|, \nu_0)$. By Corollary 4.3.4, (4.3.2) and Remark 4.4.4,

$$\begin{aligned}
&|(f \circ \varphi_j)(w) - \varphi_j(w)| \\
&\leq \frac{1}{|q'(\varphi_j(w))|}(1 + o(1) + \frac{|q'(\varphi_j(w))c_j e^{-w}|}{|p(\varphi_j(w))|}) \\
&= \frac{1}{|q'(\varphi_j(w))|}(1 + o(1) + |w|^{\lambda}|c_j e^{-w}|(1 + o(1))) \\
&\leq \frac{3}{|q'(\varphi_j(w))|} = 3|\varphi_j'(w)|
\end{aligned}$$

if $|w|$ is sufficiently large. For $\nu_0 \geq 12 + R$, with R as in Section 4.2, we obtain

$$f(\varphi_j(w)) \in \mathcal{D}(\varphi_j(w), 3|\varphi_j'(w)|) \subset \mathcal{D}\left(\varphi_j(w), \frac{\nu_0 - R}{4}|\varphi_j'(w)|\right).$$

On the other hand, by Koebe's 1/4-theorem,

$$\mathcal{S}_j \supset \varphi_j(\mathcal{D}(w, \nu_0 - R)) \supset \mathcal{D}\left(\varphi_j(w), \frac{\nu_0 - R}{4}|\varphi_j'(w)|\right),$$

whence the claim follows. □

Next, we derive an asymptotic expression for

$$h_j(w) := (q \circ f \circ \varphi_j)(w)$$

in $\mathcal{H}(\lambda, 2/|c_j|, \nu_1)$ for large $\nu_1 > 0$.

Lemma 4.6.2. *Let $j \in \{1, ..., d\}$. There exists $\nu_1 > 0$ such that*

$$h_j(w) = w - 1 + \frac{2m + 1 - d}{2d} \cdot \frac{1}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right) - c_j e^{-w}\varphi_j(w)^{d\lambda}\left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right) \tag{4.6.1}$$

as $w \to \infty$ in $\mathcal{H}(\lambda, 2/|c_j|, \nu_1)$.

Remark 4.6.3. In fact, for any $\alpha > 0$, there exists $\nu > 0$ such that h_j has an asymptotic expression of the form (4.6.1) in $\mathcal{H}(\lambda, \alpha/|c_j|, \nu)$. We need that $\alpha > 1$ so that $\mathcal{H}(\lambda, \alpha/|c_j|, \nu) \supset \mathcal{H}(\lambda, 1/|c_j|, \nu)$.

Proof of Lemma 4.6.2. By Corollary 4.3.4, we have

$$f(\varphi_j(w)) = \varphi_j(w) - \frac{\eta(w)}{q'(\varphi_j(w))},$$

where

$$\eta(w) = 1 + \frac{\lambda}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right) + c_j e^{-w}\frac{q'(\varphi_j(w))}{p(\varphi_j(w))}$$

as $w \to \infty$. Note that η is bounded in $\mathcal{H}(\lambda, 2/|c_j|, \nu_1)$. Taylor expansion of q around $\varphi_j(w)$ yields

$$\begin{aligned}
h_j(w) = q(f(\varphi_j(w))) &= \sum_{k=0}^{d} \frac{1}{k!} q^{(k)}(\varphi_j(w))(f(\varphi_j(w)) - \varphi_j(w))^k \\
&= \sum_{k=0}^{d} \frac{(-1)^k}{k!}\frac{q^{(k)}(\varphi_j(w))}{q'(\varphi_j(w))^k}\eta(w)^k \\
&= w - \eta(w) + \frac{1}{2}\frac{q''(\varphi_j(w))}{q'(\varphi_j(w))^2}\eta(w)^2 + \sum_{k=3}^{d}\frac{(-1)^k}{k!}O\left(\frac{1}{w^{k-1}}\right)\eta(w)^k \\
&= w - \eta(w) + \frac{1}{2}\frac{q''(\varphi_j(w))}{q'(\varphi_j(w))^2}\eta(w)^2 + O\left(\frac{1}{w^2}\right)
\end{aligned} \tag{4.6.2}$$

as $w \to \infty$ in $\mathcal{H}(\lambda, 2/|c_j|, \nu_1)$. Using that $\lambda = (d-1-m)/d$, we have

$$-\eta(w) = -1 + \frac{m+1-d}{d} \cdot \frac{1}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right) - c_j e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right). \tag{4.6.3}$$

Moreover,

$$\frac{q''(\varphi_j(w))}{q'(\varphi_j(w))^2} = \frac{d-1}{d} \cdot \frac{1}{w} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right)$$

and

$$\begin{aligned}\eta(w)^2 &= \left(1 + O\left(\frac{1}{w}\right) + c_j e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right)\right)^2 \\ &= 1 + O\left(\frac{1}{w}\right) + c_j e^{-w} \varphi_j(w)^{d\lambda} \cdot O(1).\end{aligned}$$

Hence

$$\frac{1}{2} \frac{q''(\varphi_j(w))}{q'(\varphi_j(w))^2} \eta(w)^2 = \frac{d-1}{2d} \frac{1}{w} + O\left(\frac{1}{|w|^{1+1/d}}\right) + c_j e^{-w} \varphi_j(w)^{d\lambda} O\left(\frac{1}{w}\right). \tag{4.6.4}$$

Combining (4.6.2), (4.6.3) and (4.6.4) yields the desired conclusion. □

For the derivative of h_j, we obtain the following.

Lemma 4.6.4. *Let $j \in \{1, ..., d\}$. There exists $\nu_2 > 0$ such that*

$$h_j'(w) = 1 + O\left(\frac{1}{|w|^{1+1/d}}\right) + c_j e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right)$$

as $w \to \infty$ in $\mathcal{H}(\lambda, 1/|c_j|, \nu_2)$.

Proof. Suppose $\nu_2 \geq \nu_1 + 1$. By Lemma 4.6.2, there are holomorphic functions a_1, a_2 satisfying $a_1(w) = O(1/|w|^{1+1/d})$ and $a_2(w) = O(1/|w|^{1/d})$ as $w \to \infty$ such that for $w \in \mathcal{H}(\lambda, 2/|c_j|, \nu_2 - 1)$, we have

$$h_j(w) = w - 1 + \frac{2m+1-d}{2d} \cdot \frac{1}{w} + a_1(w) - c_j e^{-w} \varphi_j(w)^{d\lambda}(1 + a_2(w)).$$

By Lemma 4.4.2 and Cauchy's inequality, we have $a_1'(w) = O(1/|w|^{1+1/d})$ and $a_2'(w) = O(1/|w|^{1/d})$ as $w \to \infty$ in $\mathcal{H}(\lambda, 1/|c_j|, \nu_2)$. Also,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w} e^{-w} \varphi_j(w)^{d\lambda} &= -e^{-w} \varphi_j(w)^{d\lambda} \left(1 - \frac{d\lambda}{\varphi_j(w)} \varphi_j'(w)\right) \\ &= -e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{w}\right)\right).\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}w} \left(c_j e^{-w} \varphi_j(w)^{d\lambda}(1 + a_2(w))\right) = -c_j e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right).$$

We obtain

$$\begin{aligned}h_j'(w) &= 1 - \frac{2m+1-d}{2d} \frac{1}{w^2} + a_1'(w) + c_j e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right) \\ &= 1 + O\left(\frac{1}{|w|^{1+1/d}}\right) + c_j e^{-w} \varphi_j(w)^{d\lambda} \left(1 + O\left(\frac{1}{|w|^{1/d}}\right)\right)\end{aligned}$$

as $w \to \infty$ in $\mathcal{H}(\lambda, 1/|c_j|, \nu_2)$. □

We proceed as follows. Recall that if $z_0 \in \mathcal{S}_j$ is a superattracting fixed point of f, then $q(z_0)$ lies close to the curve $\Gamma(\lambda, 1/|c_j|)$. Also, every horizontal strip of width $2\pi + \varepsilon$ that is sufficiently far from the real axis contains such an image of a superattracting fixed point. We will show that if z_0 is a superattracting fixed point of f, then $q(\mathcal{A}^*(z_0))$ contains a disk of fixed radius centred at $q(z_0)$. We then consider the backward orbit of this disk under $h_j = q \circ f \circ \varphi_j$. The function h_j is not locally invertible at $q(z_0)$; but for α slightly smaller than one, there exists a branch ψ_j of h_j^{-1} defined in $\mathcal{H}(\lambda, \alpha/|c_j|, \nu)$. If α is sufficiently close to one, then $\mathcal{H}(\lambda, \alpha/|c_j|, \nu)$ intersects the disk contained in $q(\mathcal{A}^*(z_0))$. We show that the images of this intersection under ψ_j have a certain size and are more or less evenly distributed in $\mathcal{H}(\lambda, 1/|c_j|, \nu)$. Here, the idea is that if $w \in \mathcal{H}(\lambda, 1/|c_j|, \nu)$ is not too close to the boundary, then Lemma 4.6.2 yields $h_j(w) \approx w - 1$, and hence $\psi_j(w) \approx w + 1$. See Figure 4.3 for an illustration of the aforementioned approach.

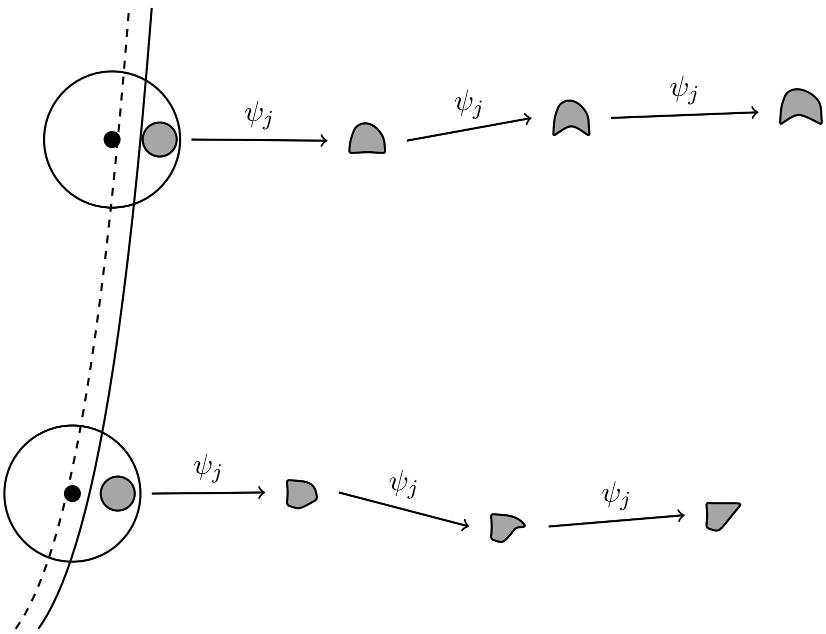


Figure 4.3: The images of superattracting fixed points of f under q, marked by black dots, lie close to the dashed line. The white disks around them are contained in the images of the corresponding attractive basins under q. To the right of the solid line, the inverse ψ_j of h_j is defined. The grey disks lie in the intersection of the images of the attractive basins and the domain of definition of ψ_j, and their forward orbits under ψ_j are contained in $q(\mathcal{F}_j)$.

Lemma 4.6.5. *There is $r_0 > 0$ such that if z_0 is a zero of g and $|z_0| \geq r_0$, then*

$$\mathcal{A}^*(z_0) \supset \mathcal{D}\left(z_0, \frac{1}{3d|z_0|^{d-1}}\right).$$

Proof. Note that g' and g'' each have only finitely many zeros. Let z_0 be a zero of g that is not a zero of g', and suppose that $\mathcal{A}^*(z_0)$ does not contain any zero of g''. Then z_0 is a superattracting fixed point of f, and there are no other critical points of f in $\mathcal{A}^*(z_0)$. Also,

$$f''(z) = \frac{g'(z)^2 g''(z) + g(z)g'(z)g'''(z) - 2g(z)g''(z)^2}{g'(z)^3},$$

and hence

$$f''(z_0) = \frac{g''(z_0)}{g'(z_0)} \neq 0.$$

By Theorem 2.4.20, there is a conformal map $\Phi : \mathcal{D}(0,1) \to \mathcal{A}^*(z_0)$ satisfying $f(\Phi(z)) = \Phi(z^2)$ and $\Phi(0) = z_0$. Differentiating the equation $f(\Phi(z)) = \Phi(z^2)$ twice yields

$$f''(\Phi(z))\Phi'(z)^2 + f'(\Phi(z))\Phi''(z) = 2\Phi'(z^2) + 4z^2\Phi''(z^2).$$

For $z = 0$, we obtain

$$f''(z_0)\Phi'(0)^2 = 2\Phi'(0)$$

and hence

$$|\Phi'(0)| = \frac{2}{|f''(z_0)|}.$$

We have

$$\begin{aligned} f''(z_0) &= \frac{g''(z_0)}{g'(z_0)} = \frac{(p(z_0)q'(z_0) + p'(z_0))e^{q(z_0)}}{p(z_0)e^{q(z_0)}} \\ &= q'(z_0) + \frac{p'(z_0)}{p(z_0)} = dz_0^{d-1}(1 + o(1)) \end{aligned}$$

as $z_0 \to \infty$. Hence, by Koebe's 1/4-theorem,

$$\mathcal{A}^*(z_0) = \Phi(\mathcal{D}(0,1)) \supset \mathcal{D}\left(z_0, \frac{1}{4}|\Phi'(0)|\right) = \mathcal{D}\left(z_0, \frac{1}{2|f''(z_0)|}\right) \supset \mathcal{D}\left(z_0, \frac{1}{3d|z_0|^{d-1}}\right)$$

if $|z_0|$ is sufficiently large. □

Corollary 4.6.6. *There exists $r_1 > 0$ such that if z_0 is a zero of g with $|z_0| \geq r_1$, then*

$$q(\mathcal{A}^*(z_0)) \supset \mathcal{D}\left(q(z_0), \frac{1}{13}\right).$$

Proof. Suppose $r_1 \geq r_0$. By Lemma 4.6.5,

$$\mathcal{A}^*(z_0) \supset \mathcal{D}\left(z_0, \frac{1}{3d|z_0|^{d-1}}\right).$$

If $|z_0|$ is sufficiently large, then q is injective in this disk, and Koebe's 1/4-theorem yields

$$q(\mathcal{A}^*(z_0)) \supset q\left(\mathcal{D}\left(z_0, \frac{1}{3d|z_0|^{d-1}}\right)\right) \supset \mathcal{D}\left(q(z_0), \frac{|q'(z_0)|}{12d|z_0|^{d-1}}\right).$$

Since $q'(z) = dz^{d-1}(1 + o(1))$ as $z \to \infty$, the claim follows. □

The next lemma deals with preimages under h_j.

Lemma 4.6.7. *Let $\alpha \in (0,1)$, $\varepsilon \in (0, 1-\alpha)$ and $j \in \{1, ..., d\}$. There exists $\nu_3 > 0$ such that for each $w_0 \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_3)$, there is a unique $w \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_3 - 1)$ with $h_j(w) = w_0$. More precisely, $w \in \mathcal{D}(w_0 + 1, \alpha + \varepsilon)$.*

Proof. Let $w \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_3 - 1)$. By Lemma 4.6.2 and Remark 4.4.4,

$$\begin{aligned} |h_j(w) - (w-1)| &\leq o(1) + |c_j e^{-w}||w|^\lambda(1 + o(1)) \\ &\leq o(1) + \alpha(1 + o(1)) < \alpha + \varepsilon \end{aligned}$$

if ν_3 and hence $|w|$ is sufficiently large. If $h_j(w) = w_0$, we deduce

$$|w - (w_0 + 1)| = |w_0 - (w - 1)| = |h_j(w) - (w - 1)| < \alpha + \varepsilon;$$

that is, $w \in \mathcal{D}(w_0 + 1, \alpha + \varepsilon)$. On the other hand, Lemma 4.4.2 yields that

$$\overline{\mathcal{D}(w_0 + 1, \alpha + \varepsilon)} \subset \mathcal{H}(\lambda, \alpha/|c_j|, \nu_3 - 1)$$

if ν_3 is sufficiently large. Thus for $w \in \partial\mathcal{D}(w_0 + 1, \alpha + \varepsilon)$, we have

$$|(h_j(w) - w_0) - (w - 1 - w_0)| = |h_j(w) - (w - 1)| < \alpha + \varepsilon = |w - 1 - w_0|.$$

By Rouché's theorem, there is a unique $w \in \mathcal{D}(w_0 + 1, \alpha + \varepsilon)$ satisfying $h_j(w) = w_0$. $\square$

By Lemma 4.6.7, there is a subset $\mathcal{H}_j \subset \mathcal{H}(\lambda, \alpha/|c_j|, \nu_3 - 1)$ such that h_j maps $\mathcal{H}_j$ conformally onto $\mathcal{H}(\lambda, \alpha/|c_j|, \nu_3)$. Let $\psi_j : \mathcal{H}(\lambda, \alpha/|c_j|, \nu_3) \to \mathcal{H}_j$ be the corresponding inverse function. Parts (i) and (iii) of the next lemma yield that if $|\operatorname{Im} w|$ is sufficiently large, then all iterates $\psi_j^n(w)$ are defined and tend to infinity in a horizontal strip whose width is bounded independent of w. Parts (ii) and (iv) are used to prove (iii), and will be reused to estimate $(\psi_j^n)'$.

Lemma 4.6.8. *Let $\alpha \in (0, 1)$, $\varepsilon \in (0, 1-\alpha)$ and $j \in \{1, ..., d\}$. Set $\delta := 1-\alpha-\varepsilon$. Then there exist $\nu_4 > 0$ and $C > 0$ such that $\psi_j^n(w)$ is defined for all $w \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_4)$ and all $n \in \mathbb{N}$, and satisfies*

(i) $\operatorname{Re} \psi_j^n(w) \geq \operatorname{Re} w + n\delta$;

(ii) $|\psi_j^n(w)| \geq \max\{n, |w|\} \cdot \delta/4$;

(iii) $|\operatorname{Im} \psi_j^n(w) - \operatorname{Im} w| \leq C$;

(iv) $e^{-\operatorname{Re} \psi_j^n(w)} |\psi_j^n(w)|^\lambda = O(e^{-n\delta/2})$.

Proof. First note that if $\psi_j^n(w)$ is defined, then Lemma 4.6.7 yields that $\psi_j^k(w) \in \mathcal{D}(\psi_j^{k-1}(w) + 1, \alpha + \varepsilon)$ for all $k \in \{1, ..., n\}$, and hence

$$\operatorname{Re} \psi_j^n(w) \geq \operatorname{Re} w + n\delta.$$

So $\psi_j^n(w)$ satisfies (i). Also, for $n \leq |w|/2$,

$$|\psi_j^n(w)| \geq |w| - n(1 + \alpha + \varepsilon) \geq |w| - \frac{|w|}{2}(1 + \alpha + \varepsilon) = |w|\frac{\delta}{2} \geq n\delta.$$

For $n > |w|/2$,

$$|\psi_j^n(w)| \geq \operatorname{Re} \psi_j^n(w) \geq \operatorname{Re} w + n\delta \geq \lambda \log |w| - \log \frac{\alpha}{|c_j|} + n\delta \geq \frac{n\delta}{2} \geq \frac{|w|\delta}{4}$$

if $|w|$ and hence also n is sufficiently large. In particular, $\psi_j^n(w)$ satisfies (ii).

Let

$$n_w := \lfloor |w| \rfloor = \max\{k \in \mathbb{Z} : \ k \leq |w|\}.$$

We show by induction that if $w \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_4)$ for large $\nu_4 > 0$, then $\psi_j^n(w)$ is defined for all $n \in \mathbb{N}$ and

$$|\operatorname{Im} \psi_j^n(w) - \operatorname{Im} w| \leq C' \left(\min\left\{\frac{n}{|w|}, 1\right\} + n_w \sum_{k=n_w}^{n} \frac{1}{k^2} + \sum_{k=1}^{n} \frac{1}{k^{1+1/d}} + \sum_{k=1}^{n} e^{-k\delta/2} \right) \tag{4.6.5}$$

for some constant C' that does not depend on w or n. Note that by Lemma 2.6.2,

$$\begin{aligned}
&C'\left(\min\left\{\frac{n}{|w|},1\right\}+n_w\sum_{k=n_w}^{n}\frac{1}{k^2}+\sum_{k=1}^{n}\frac{1}{k^{1+1/d}}+\sum_{k=1}^{n}e^{-k\delta/2}\right)\\
\leq &C'\left(3+\sum_{k=1}^{\infty}\frac{1}{k^{1+1/d}}+\sum_{k=1}^{\infty}e^{-k\delta/2}\right)=: C<\infty.
\end{aligned}$$

So (4.6.5) implies (iii). Clearly, (4.6.5) is true for $n=0$. Suppose that (4.6.5) holds with n replaced by $n-1$. By Lemma 4.6.7, $\psi_j^n(w)$ is defined if and only if $|\operatorname{Im}\psi_j^{n-1}(w)|>\nu_3$. This is satisfied if $|\operatorname{Im} w|>\nu_3+C$. By Lemma 4.6.2,

$$\begin{aligned}
&|\operatorname{Im}\psi_j^n(w)-\operatorname{Im}\psi_j^{n-1}(w)| = |\operatorname{Im}\psi_j^n(w)-\operatorname{Im}h_j(\psi_j^n(w))|\\
\leq &\left|\frac{2m+1-d}{2d}\operatorname{Im}\left(\frac{1}{\psi_j^n(w)}\right)\right|+O\left(\frac{1}{|\psi_j^n(w)|^{1+1/d}}\right)+2|c_j|e^{-\operatorname{Re}\psi_j^n(w)}|\psi_j^n(w)|^\lambda
\end{aligned}$$

if $|w|$ is sufficiently large. By (ii),

$$\frac{1}{|\psi_j^n(w)|^{1+1/d}}=O\left(\frac{1}{n^{1+1/d}}\right).$$

If $n\leq|w|$, then we estimate the first summand by

$$\left|\operatorname{Im}\left(\frac{1}{\psi_j^n(w)}\right)\right|\leq\frac{1}{|\psi_j^n(w)|}=O\left(\frac{1}{|w|}\right).$$

If $n>|w|$, then by Lemma 4.6.7, (ii) and the induction hypothesis,

$$\begin{aligned}
\left|\operatorname{Im}\left(\frac{1}{\psi_j^n(w)}\right)\right| &= \frac{|\operatorname{Im}\psi_j^n(w)|}{|\psi_j^n(w)|^2}\leq\frac{|\operatorname{Im}\psi_j^{n-1}(w)|+\alpha+\varepsilon}{|\psi_j^n(w)|^2}\\
&\leq\frac{16(|\operatorname{Im} w|+C+\alpha+\varepsilon)}{\delta^2n^2}\leq\frac{17|w|}{\delta^2n^2}=n_w\cdot O\left(\frac{1}{n^2}\right),
\end{aligned}$$

provided $|w|$ is sufficiently large.

Moreover, if $\lambda\geq 0$, then by (i), Lemma 4.6.7 and Remark 4.4.4,

$$\begin{aligned}
|c_j|e^{-\operatorname{Re}\psi_j^n(w)}|\psi_j^n(w)|^\lambda &\leq |c_j|e^{-\operatorname{Re} w}(|w|+n(1+\alpha+\varepsilon))^\lambda e^{-n\delta}\\
&\leq\alpha|w|^{-\lambda}(|w|+n(1+\alpha+\varepsilon))^\lambda e^{-n\delta}\\
&=\alpha\left(1+\frac{n}{|w|}(1+\alpha+\varepsilon)\right)^\lambda e^{-n\delta}\\
&\leq\alpha(1+n(1+\alpha+\varepsilon))^\lambda e^{-n\delta}=O(e^{-n\delta/2}),
\end{aligned}$$

provided $|w|\geq 1$. If $\lambda<0$, then by (i), (ii) and Remark 4.4.4,

$$|c_j|e^{-\operatorname{Re}\psi_j^n(w)}|\psi_j^n(w)|^\lambda\leq\left(\frac{\delta}{4}\right)^\lambda|w|^\lambda|c_j|e^{-\operatorname{Re} w}e^{-n\delta}\leq\left(\frac{\delta}{4}\right)^\lambda\alpha e^{-n\delta}.$$

In particular, (iv) is satisfied. Also, if $n\leq|w|$, then

$$\operatorname{Im}\psi_j^n(w)-\operatorname{Im}\psi_j^{n-1}(w)=O\left(\frac{1}{|w|}\right)+O\left(\frac{1}{n^{1+1/d}}\right)+O(e^{-n\delta/2});$$

and if $n > |w|$, then

$$\operatorname{Im} \psi_j^n(w) - \operatorname{Im} \psi_j^{n-1}(w) = n_w O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^{1+1/d}}\right) + O(e^{-n\delta/2}).$$

Thus $\psi_j^n(w)$ satisfies (4.6.5) and hence also (iii). $\square$

Next, we estimate $(\psi_j^n)'$.

Lemma 4.6.9. *Let $\alpha \in (0,1)$ and $j \in \{1, ..., d\}$. There are $\nu_5 > 0$ and $B > 0$ such that*

$$|(\psi_j^n)'(w)| \geq B$$

for all $w \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_5)$ and all $n \in \mathbb{N}$.

Proof. We have

$$(\psi_j^n)' = \prod_{k=0}^{n-1} \psi_j' \circ \psi_j^k = \frac{1}{\prod_{k=0}^{n-1} h_j' \circ \psi_j^{k+1}} = \frac{1}{\prod_{k=1}^{n} h_j' \circ \psi_j^k}.$$

By Lemma 4.6.4 and Lemma 4.6.8,

$$\begin{aligned} &|h_j'(\psi_j^k(w))| \\ &\leq 1 + O\left(\frac{1}{|\psi_j^k(w)|^{1+1/d}}\right) + |c_j| e^{-\operatorname{Re} \psi_j^k(w)} |\psi_j^k(w)|^\lambda \left(1 + O\left(\frac{1}{|\psi_j^k(w)|^{1/d}}\right)\right) \\ &\leq 1 + O\left(\frac{1}{k^{1+1/d}}\right) + O\left(e^{-k\delta/2}\right). \end{aligned}$$

Since the infinite product $\prod_{k=1}^\infty (1 + O(1/k^{1+1/d}) + O(e^{-k\delta/2}))$ converges, this implies the desired conclusion. $\square$

Recall that $\mathcal{F}_j = \mathcal{F}(f) \cap \mathcal{S}_j$.

Lemma 4.6.10. *For $j \in \{1, ..., d\}$ and $k \in \mathbb{Z}$, let $v_{j,k}$ be as in Lemma 4.5.5 and $w_{j,k} := v_{j,k} + 1/26$. There are $k_0, \vartheta > 0$ such that if $|k| \geq k_0$, then $\mathcal{D}(\psi_j^n(w_{j,k}), \vartheta) \subset q(\mathcal{F}_j)$ for all $n \in \mathbb{N}$.*

Remark 4.6.11. For large $|k|$, the point $v_{j,k}$ is close to $q(z_0)$ for some attracting fixed point z_0 of f. The function ψ_j is not defined in $q(z_0)$ and possibly $v_{j,k}$. Therefore, we introduce the point $w_{j,k}$, which lies in the intersection of the domain of definition of ψ_j and $q(\mathcal{A}^*(z_0))$ if $|k|$ is large.

Proof of Lemma 4.6.10. By Lemma 4.5.5, there is a zero z_0 of g satisfying $q(z_0) = v_{j,k} + o(1)$. Thus, $w_{j,k} = q(z_0) + 1/26 + o(1)$. If $|k|$ is sufficiently large, we get

$$\mathcal{D}\left(w_{j,k}, \frac{1}{27}\right) \subset \mathcal{D}\left(q(z_0) + \frac{1}{26}, \frac{1}{26}\right) \subset \mathcal{D}\left(q(z_0), \frac{1}{13}\right).$$

By Corollary 4.6.6, this yields

$$\mathcal{D}\left(w_{j,k}, \frac{1}{27}\right) \subset q(\mathcal{A}^*(z_0)).$$

Let $\exp(-1/2(1/26 - 1/27)) < \alpha < 1$. Then

$$2 \log \frac{1}{\alpha} < \frac{1}{26} - \frac{1}{27}. \tag{4.6.6}$$

Since $v_{j,k} \in \Gamma(\lambda, 1/|c_j|)$, by (4.6.6) and Lemma 4.4.2 (iv) and (iii), we have

$$\mathcal{D}\left(w_{j,k}, \frac{1}{27}\right) \subset \mathcal{H}\left(\lambda, \frac{\alpha}{|c_j|}, \nu_5\right)$$

if $|k|$ is sufficiently large. By Koebe's 1/4-theorem and Lemma 4.6.9,

$$\psi_j^n\left(\mathcal{D}\left(w_{j,k}, \frac{1}{27}\right)\right) \supset \mathcal{D}\left(\psi_j^n(w_{j,k}), \frac{|(\psi_j^n)'(w_{j,k})|}{4 \cdot 27}\right) \supset \mathcal{D}\left(\psi_j^n(w_{j,k}), \frac{B}{4 \cdot 27}\right).$$

For $\vartheta := B/(4 \cdot 27)$, we thus have

$$h_j^n(\mathcal{D}(\psi_j^n(w_{j,k}), \vartheta)) \subset \mathcal{D}\left(w_{j,k}, \frac{1}{27}\right) \subset q(\mathcal{F}_j).$$

Using that by Lemma 4.6.1,

$$h_j^n(w) = (q \circ f \circ \varphi_j)^n(w) = (q \circ f^n \circ \varphi_j)(w)$$

for $w \in \mathcal{H}(\lambda, \alpha/|c_j|, \nu_0)$, we deduce that

$$\mathcal{D}(\psi_j^n(w_{j,k}), \vartheta) \subset q(\mathcal{F}_j)$$

if $|k|$ is sufficiently large. □

Remark 4.6.12. Let $z_0, v_{j,k}, w_{j,k}$ be as in Lemma 4.6.10 and its proof. Similar as in [Jan96, p. 53], a more precise description of $q(\mathcal{A}^*(z_0))$ can be obtained as follows. By Lemma 4.3.1 and the proof of Lemma 4.5.5, we have

$$g(\varphi_j(v_{j,k})) = c_j + \varphi_j(v_{j,k})^{-d\lambda}(1 + o(1))e^{v_{j,k}} = o(1)$$

as $|k| \to \infty$. Using Lemma 4.6.2, one can deduce that

$$h_j(w_{j,k}) = v_{j,k} + \frac{1}{26} - 1 + e^{-1/26} + o(1)$$

as $|k| \to \infty$. Because $0 < 1/26 - 1 + e^{-1/26} < 1/26$, this implies that $h_j(w_{j,k}) \in \mathcal{H}(\lambda, \alpha'/|c_j|, \nu_5) \cap q(\mathcal{A}^*(z_0))$ for some $\alpha' \in (0,1)$ if $|k|$ is sufficiently large. Let σ be the straight line connecting $w_{j,k}$ and $h_j(w_{j,k})$. Then $\tau := \bigcup_{n=0}^{\infty} \psi_j^n(\sigma)$ is a curve entirely contained in $q(\mathcal{A}^*(z_0))$. By Lemma 4.6.8, we have $\operatorname{Re}\tau \to +\infty$ and τ is contained in a horizontal strip whose width is bounded independent of k. By the same arguments as in the proof of Lemma 4.6.10, there exists $\vartheta' > 0$ such that

$$\{w \in \mathbb{C} : \operatorname{dist}(w, \tau) < \vartheta'\} \subset q(\mathcal{A}^*(z_0)).$$

The final result of this section says that the density of $q(\mathcal{F}_j)$ in large rectangles contained in $\mathcal{H}(\lambda, 1/|c_j|, \nu)$ is bounded away from zero.

Lemma 4.6.13. *There are $D_0, \nu_0, \eta_0 > 0$ such that for all $j \in \{1, ..., d\}$ and any rectangle $S \subset \mathcal{H}(\lambda, 1/|c_j|, \nu_0)$ with sides parallel to the real and imaginary axis whose vertical and horizontal side lengths are both at least D_0, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), S) \geq \eta_0.$$

Proof. First suppose that S is of the form

$$S = \{w \in \mathbb{C} : x_1 \leq \operatorname{Re} w \leq x_2 \text{ and } y_1 \leq \operatorname{Im} w \leq y_2\}, \tag{4.6.7}$$

where

$$2\pi + 2(C + \vartheta) \leq x_2 - x_1, y_2 - y_1 \leq 2(2\pi + 2(C + \vartheta)) \tag{4.6.8}$$

with C as in Lemma 4.6.8 and ϑ as in Lemma 4.6.10. Let $v_{j,k}$ be as in Lemma 4.5.5 and $w_{j,k} = v_{j,k} + 1/26$. There is $k \in \mathbb{Z}$ such that $y_1 + C + \vartheta \leq \operatorname{Im} w_{j,k} = \operatorname{Im} v_{j,k} \leq y_2 - C - \vartheta$. Also, by Lemma 4.6.7, there exists $n \in \mathbb{N}$ such that $x_1 + \vartheta < \operatorname{Re} \psi_j^n(w_{j,k}) < x_2 - \vartheta$. By Lemma 4.6.8, we have $y_1 + \vartheta \leq \operatorname{Im} \psi_j^n(w_{j,k}) \leq y_2 - \vartheta$. Thus

$$\mathcal{D}(\psi_j^n(w_{j,k}), \vartheta) \subset S.$$

Also, by Lemma 4.6.10,

$$\mathcal{D}(\psi_j^n(w_{j,k}), \vartheta) \subset q(\mathcal{F}_j).$$

Hence

$$\operatorname{dens}(q(\mathcal{F}_j), S) \geq \frac{\operatorname{meas}(\mathcal{D}(\psi_j^n(w_{j,k}), \vartheta))}{\operatorname{meas}(S)} \geq \frac{\pi \vartheta^2}{4(2\pi + 2(C + \vartheta))^2} =: \eta_0.$$

If $S \subset \mathcal{H}(\lambda, 1/|c_j|, \nu_0)$ is an arbitrary rectangle whose horizontal and vertical side length both exceed $D_0 := 2\pi + 2(C + \vartheta)$, then S can be written as the union of rectangles of the form (4.6.7) that satisfy (4.6.8) and have pairwise disjoint interior, whence the claim follows. □

4.7 The set $q(\mathcal{F}(f))$: second part

In this section we investigate the density of $q(\mathcal{F}(f))$ in subsets of $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ for small $\alpha_1 > 0$ and large $\beta_1 > 0$.

We begin with an approximation for h_j.

Lemma 4.7.1. *For any $\varepsilon > 0$, there are $\alpha_1, \beta_1, \nu > 0$ such that for all $j \in \{1, ..., d\}$ and all $w \in \mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$, we have*

$$\left| \frac{h_j(w) - w}{-c_j e^{-w} \varphi_j(w)^{d\lambda}} - 1 \right| < \varepsilon.$$

Proof. Suppose without loss of generality that $\varepsilon < 1$. Let $\beta_1 > 4d/\varepsilon$ and $\alpha_1 < \varepsilon/(16(d-1)!)$. Taylor expansion of q around $\varphi_j(w)$ yields

$$h_j(w) = q(f(\varphi_j(w))) = w + \sum_{k=1}^{d} \frac{q^{(k)}(\varphi_j(w))}{k!} (f(\varphi_j(w)) - \varphi_j(w))^k.$$

Thus

$$\begin{aligned} &\frac{h_j(w) - w}{-c_j e^{-w} \varphi_j(w)^{d\lambda}} - 1 \\ &= \frac{q'(\varphi_j(w))(f(\varphi_j(w)) - \varphi_j(w))}{-c_j e^{-w} \varphi_j(w)^{d\lambda}} - 1 + \sum_{k=2}^{d} \frac{q^{(k)}(\varphi_j(w))}{k!} \frac{(f(\varphi_j(w)) - \varphi_j(w))^k}{-c_j e^{-w} \varphi_j(w)^{d\lambda}}. \end{aligned} \tag{4.7.1}$$

By Corollary 4.3.4,

$$f(\varphi_j(w)) = \varphi_j(w) - \frac{1}{q'(\varphi_j(w))} (1 + o(1) + c_j e^{-w} \varphi_j(w)^{d\lambda} (1 + o(1))) \tag{4.7.2}$$

as $w \to \infty$. Hence

$$\frac{q'(\varphi_j(w))(f(\varphi_j(w)) - \varphi_j(w))}{-c_j e^{-w}\varphi_j(w)^{d\lambda}} - 1 = \frac{1 + o(1)}{c_j e^{-w}\varphi_j(w)^{d\lambda}} + o(1). \tag{4.7.3}$$

For $w \in \mathbb{C} \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ with $|\operatorname{Im} w| \geq \nu$, we have

$$|c_j e^{-w}\varphi_j(w)^{d\lambda}| \geq \frac{\beta_1}{2}$$

if ν is sufficiently large. Inserting this inequality into (4.7.3) yields

$$\left|\frac{q'(\varphi_j(w))(f(\varphi_j(w)) - \varphi_j(w))}{-c_j e^{-w}\varphi_j(w)^{d\lambda}} - 1\right| \leq \frac{3}{\beta_1} + o(1) < \frac{3\varepsilon}{4d} + o(1) < \frac{\varepsilon}{d} \tag{4.7.4}$$

if $|w|$ is sufficiently large.

Also, for $w \in \mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu)$, we have

$$|c_j e^{-w}\varphi_j(w)^{d(\lambda-1)}| \leq 2\alpha_1$$

if ν is sufficiently large. By (4.7.2), this implies

$$|f(\varphi_j(w)) - \varphi_j(w)| \leq \frac{1}{|q'(\varphi_j(w))|}(1 + o(1) + 3\alpha_1\varphi_j(w)^d) \leq \frac{4}{d}\alpha_1|\varphi_j(w)| \tag{4.7.5}$$

if $|w|$ is sufficiently large. For $k \geq 2$, equations (4.7.4) and (4.7.5) yield

$$\begin{aligned}
&\left|\frac{q^{(k)}(\varphi_j(w))}{k!} \cdot \frac{(f(\varphi_j(w)) - \varphi_j(w))^k}{-c_j e^{-w}\varphi_j(w)^{d\lambda}}\right| \\
&= \left|\frac{q'(\varphi_j(w))(f(\varphi_j(w)) - \varphi_j(w))}{-c_j e^{-w}\varphi_j(w)^{d\lambda}}\right| \cdot \left|\frac{q^{(k)}(\varphi_j(w))}{k! q'(\varphi_j(w))}\right| \cdot |f(\varphi_j(w)) - \varphi_j(w)|^{k-1} \\
&\leq \left(1 + \frac{\varepsilon}{d}\right) \cdot \left|\frac{q^{(k)}(\varphi_j(w))}{k! q'(\varphi_j(w))}\right| \cdot \left(\frac{4}{d}\alpha_1|\varphi_j(w)|\right)^{k-1} \\
&\leq \left(1 + \frac{\varepsilon}{d}\right)\binom{d}{k}\frac{2}{d}|\varphi_j(w)|^{-k+1}\left(\frac{4}{d}\alpha_1|\varphi_j(w)|\right)^{k-1} \\
&= \left(1 + \frac{\varepsilon}{d}\right)\binom{d}{k}\frac{2}{d}\left(\frac{4\alpha_1}{d}\right)^{k-1} < 2d! \cdot \frac{2}{d}\left(\frac{\varepsilon}{4d!}\right)^{k-1} \\
&\leq 2d! \cdot \frac{2}{d} \cdot \frac{\varepsilon}{4d!} = \frac{\varepsilon}{d}
\end{aligned} \tag{4.7.6}$$

if $|w|$ is sufficiently large. Inserting (4.7.4) and (4.7.6) into (4.7.1) yields the desired conclusion. □

We proceed as follows. First, we show that h_j maps the intersection of certain horizontal strips with $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ into $\mathcal{H}(\lambda, 1/c^*, \nu)$, where $c^* = \max_l |c_l|$. The idea is that if $\operatorname{Im} w$ lies in certain intervals, then the argument of $-c_j e^{-w}\varphi_j(w)^{d\lambda}$ is small; and since $h_j(w) \approx w - c_j e^{-w}\varphi_j(w)^{d\lambda}$ by Lemma 4.7.1, one can deduce that $\operatorname{Re} h_j(w)$ is large. By Section 4.6, the density of $q(\mathcal{F}(f))$ in large rectangles in $\mathcal{H}(\lambda, 1/c^*, \nu)$ is bounded away from zero. Together with the invariance of $\mathcal{F}(f)$ under f, we deduce that the density of $q(\mathcal{F}(f))$ in large bounded subsets of $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ is bounded away from zero.

The next lemma describes the mapping behaviour of h_j in certain horizontal strips in $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$. For $j \in \{1, ..., d\}$ and $n \in \mathbb{Z}$, let

$$y_n^j := \begin{cases} \arg(-c_j) + \lambda(\pi/2 + 2\pi(j-1)) + 2n\pi & \text{if } n \geq 0 \\ \arg(-c_j) + \lambda(-\pi/2 + 2\pi j) + 2n\pi & \text{if } n < 0. \end{cases}$$

Lemma 4.7.2. *Let $j \in \{1, ..., d\}$. For any $\varepsilon \in (0, \pi/4)$, there are $\alpha_1, \beta_1, \nu > 0$ such that the following holds.*

Suppose that w lies in the closure of $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ and there exists $n \in \mathbb{Z}$ with $|\operatorname{Im} w - y_n^j| \leq \pi/4$. Let $\beta \geq \beta_1$ such that $w \in \Gamma(\lambda, \beta/|c_j|)$, and set $\theta := \operatorname{Im} w - y_n^j$. Then

$$|h_j(w) - w| \leq (1+\varepsilon)\beta,$$

$$(1-\varepsilon)\beta\cos(|\theta| + \varepsilon) \leq \operatorname{Re}(h_j(w) - w) \leq (1+\varepsilon)\beta$$

and

$$(1-\varepsilon)\beta\sin(|\theta| - \varepsilon) \leq |\operatorname{Im}(h_j(w) - w)| \leq (1+\varepsilon)\beta\sin(|\theta| + \varepsilon).$$

Proof. Suppose that α_1, β_1, ν are chosen such that the conclusion of Lemma 4.7.1 holds with ε replaced by $\varepsilon/2$; that is,

$$\left| \frac{h_j(w) - w}{-c_j e^{-w} \varphi_j(w)^{d\lambda}} - 1 \right| < \frac{\varepsilon}{2}. \tag{4.7.7}$$

Then

$$\left(1 - \frac{\varepsilon}{2}\right) |c_j e^{-w} \varphi_j(w)^{d\lambda}| < |h_j(w) - w| < \left(1 + \frac{\varepsilon}{2}\right) |c_j e^{-w} \varphi_j(w)^{d\lambda}|.$$

Since $w \in \Gamma(\lambda, \beta/|c_j|)$, this implies

$$(1-\varepsilon)\beta \leq |h_j(w) - w| \leq (1+\varepsilon)\beta$$

if ν is sufficiently large. Also, by (4.7.7),

$$\left| \arg\left(\frac{h_j(w) - w}{-c_j e^{-w} \varphi_j(w)^{d\lambda}} \right) \right| \leq \arcsin\left(\frac{\varepsilon}{2}\right) \leq \frac{\pi}{4}\varepsilon. \tag{4.7.8}$$

We have

$$\arg w = \arg q(\varphi_j(w)) \equiv \arg(\varphi_j(w)^d(1 + o(1))) \equiv d \arg \varphi_j(w) + o(1) \mod 2\pi$$

as $w \to \infty$. Since w lies in the closure of $\mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$, Lemma 4.4.5 and the subsequent remark yield

$$\arg \varphi_j(w) \equiv \begin{cases} \pi/(2d) + 2\pi(j-1)/d + o(1) & \text{if } n > 0 \\ -\pi/(2d) + 2\pi j/d + o(1) & \text{if } n < 0 \end{cases} \mod 2\pi.$$

Thus

$$\begin{aligned} \arg\left(-c_j e^{-w} \varphi_j(w)^{d\lambda}\right) &\equiv \arg(-c_j) - \operatorname{Im} w + d\lambda \arg(\varphi_j(w)) \\ &\equiv -\theta - 2n\pi + o(1) \equiv -\theta + o(1) \mod 2\pi. \end{aligned}$$

By (4.7.8), this implies

$$|\theta| - \varepsilon \leq |\arg(h_j(w) - w)| \leq |\theta| + \varepsilon$$

if $|w|$ is sufficiently large. We deduce

$$\begin{aligned} \operatorname{Re}(h_j(w) - w) &\leq |h_j(w) - w| \leq (1+\varepsilon)\beta, \\ \operatorname{Re}(h_j(w) - w) &= |h_j(w) - w| \cos(\arg(h_j(w) - w)) \geq (1-\varepsilon)\beta\cos(|\theta| + \varepsilon), \\ |\operatorname{Im}(h_j(w) - w)| &= |h_j(w) - w| \cdot |\sin(\arg(h_j(w) - w))| \leq (1+\varepsilon)\beta\sin(|\theta| + \varepsilon), \\ |\operatorname{Im}(h_j(w) - w)| &= |h_j(w) - w| \cdot |\sin(\arg(h_j(w) - w))| \geq (1-\varepsilon)\beta\sin(|\theta| - \varepsilon). \end{aligned}$$ ☐

Set
$$c^* := \max_{1\le l\le d} |c_l|.$$
Let ν_0 be as in Lemma 4.6.13. The following lemma says that h_j maps the intersection of $\mathcal{H}(\lambda-1,\alpha_1/|c_j|,\nu)\setminus\mathcal{H}(\lambda,\beta_1/|c_j|,\nu)$ with certain horizontal strips into $\mathcal{H}(\lambda,1/c^*,\nu_0)$.

Lemma 4.7.3. *There are $\alpha_1,\beta_1,\nu>0$ such that for all $j\in\{1,...,d\}$, all $n\in\mathbb{Z}$ and all w in the closure of $\mathcal{H}(\lambda-1,\alpha_1/|c_j|,\nu)\setminus\mathcal{H}(\lambda,\beta_1/|c_j|,\nu)$ satisfying $|\operatorname{Im} w - y_n^j|\le \pi/4$, we have $h_j(w)\in\mathcal{H}(\lambda,1/c^*,\nu_0)$.*

Proof. We have $w\in\Gamma(\lambda,\beta/|c_j|)$ for some $\beta\ge\beta_1$. Since also $w\in\mathcal{H}(\lambda-1,\alpha_1/|c_j|,\nu)$, we have
$$\lambda\log|w|-\log\frac{\beta}{|c_j|}=\operatorname{Re} w\ge(\lambda-1)\log|w|-\log\frac{\alpha_1}{|c_j|}$$
and hence
$$\beta\le\alpha_1|w|. \tag{4.7.9}$$
Let $\theta:=\operatorname{Im} w-y_n^j$. Suppose $\alpha_1\le 1/3$, and assume that α_1,β_1,ν are chosen such that the conclusion of Lemma 4.7.2 holds for $\varepsilon=1/2$. Then
$$|\operatorname{Im} h_j(w)-\operatorname{Im} w|\le\frac{3}{2}\beta\sin\left(|\theta|+\frac{1}{2}\right)\le\frac{3}{2}\beta.$$
By Lemma 4.4.5 and (4.7.9), this yields
$$\begin{aligned}|\operatorname{Im} h_j(w)|&\ge|\operatorname{Im} w|-\frac{3}{2}\beta=(1+o(1))|w|-\frac{3}{2}\beta\\&\ge(1+o(1))|w|-\frac{3}{2}\alpha_1|w|\ge\left(\frac{1}{2}+o(1)\right)|w|>\nu_0,\end{aligned}$$
provided ν and hence $|w|$ is sufficiently large.

Also, by Lemma 4.7.2 and (4.7.9),
$$|h_j(w)-w|\le\frac{3}{2}\beta\le\frac{3}{2}\alpha_1|w|\le\frac{|w|}{2}.$$
We obtain
$$\frac{1}{2}|w|\le|h_j(w)|\le 2|w|$$
and hence
$$\frac{1}{2}|h_j(w)|\le|w|\le 2|h_j(w)|. \tag{4.7.10}$$
By Lemma 4.7.2 and (4.7.10),
$$\begin{aligned}\operatorname{Re} h_j(w)&\ge\operatorname{Re} w+\frac{1}{2}\beta\cos\left(\frac{\pi}{4}+\frac{1}{2}\right)\\&=\lambda\log|w|-\log\beta+\frac{1}{2}\beta\cos\left(\frac{\pi}{4}+\frac{1}{2}\right)\\&\ge\lambda\log|h_j(w)|-|\lambda|\log 2-\log\beta+\frac{1}{2}\beta\cos\left(\frac{\pi}{4}+\frac{1}{2}\right)\\&\ge\lambda\log|h_j(w)|-\log\frac{1}{c^*}\end{aligned}$$
if β_1 and hence β is sufficiently large. Thus $h_j(w)\in\mathcal{H}(\lambda,1/c^*,\nu_0)$. $\square$

Let us now define several sets. We start with subsets $\mathcal{Q}^j_{n,k}, \tilde{\mathcal{Q}}^j_{n,k} \subset \mathbb{C}\backslash\mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ for $j \in \{1, ..., d\}$, $k \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Fix

$$\theta_1 \in \left(0, \frac{1}{6\pi} \arccos\left(\frac{5}{6}\right)\right).$$

For $j \in \{1, ..., d\}$, $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, let $\mathcal{Q}^j_{n,k}$ be the set of all

$$w \in \mathcal{H}\left(\lambda, \frac{2^{k+1}\beta_1}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda, \frac{2^k\beta_1}{|c_j|}, \nu\right)$$

such that

$$|\operatorname{Im} w - y^j_n| \le \theta_1.$$

Also, let $\tilde{\mathcal{Q}}^j_{n,k}$ be the set of all

$$w \in \mathcal{H}\left(\lambda, \frac{2^{k+2}\beta_1}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda, \frac{2^{k-1}\beta_1}{|c_j|}, \nu\right)$$

such that

$$|\operatorname{Im} w - y^j_n| \le 5\pi\theta_1.$$

See Figure 4.4 for an illustration of these sets. Note that $\mathcal{Q}^j_{n,k} \subset \tilde{\mathcal{Q}}^j_{n,k}$. If $\tilde{\mathcal{Q}}^j_{n,k} \subset \mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu)$, then Lemma 4.7.3 yields $h_j(\tilde{\mathcal{Q}}^j_{n,k}) \subset \mathcal{H}(\lambda, 1/c^*, \nu)$.

Define further rectangles $\mathcal{R}^j_{n,k}, \tilde{\mathcal{R}}^j_{n,k}$ as follows. Let $\mathcal{R}^j_{n,k}$ be the set of all $v \in \mathbb{C}$ satisfying

$$\frac{3}{4}2^k\beta_1 < \operatorname{Re} v - \lambda \log|n| < \frac{5}{2}2^k\beta_1$$

and

$$|\operatorname{Im} v - y^j_n| < 3 \cdot 2^k\beta_1\theta_1.$$

Also, let $\tilde{\mathcal{R}}^j_{n,k}$ be the rectangle containing all $v \in \mathbb{C}$ satisfying

$$\frac{5}{8}2^k\beta_1 < \operatorname{Re} v - \lambda \log|n| < 3 \cdot 2^k\beta_1$$

and

$$|\operatorname{Im} v - y^j_n| < 4 \cdot 2^k\beta_1\theta_1.$$

Note that $\mathcal{R}^j_{n,k} \subset \tilde{\mathcal{R}}^j_{n,k}$.

Lemma 4.7.4. *There are $\alpha_1, \beta_1, \nu > 0$ such that the following holds.*

If $j \in \{1, ..., d\}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ are such that $\tilde{\mathcal{Q}}^j_{n,k} \subset \mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu)$, then $h_j(\mathcal{Q}^j_{n,k}) \subset \mathcal{R}^j_{n,k}$ and $h_j(\tilde{\mathcal{Q}}^j_{n,k}) \supset \tilde{\mathcal{R}}^j_{n,k}$.

Proof. Let α_1, β_1, ν be chosen such that the conclusion of Lemma 4.7.2 holds for $\varepsilon = \theta_1/3$. Also suppose

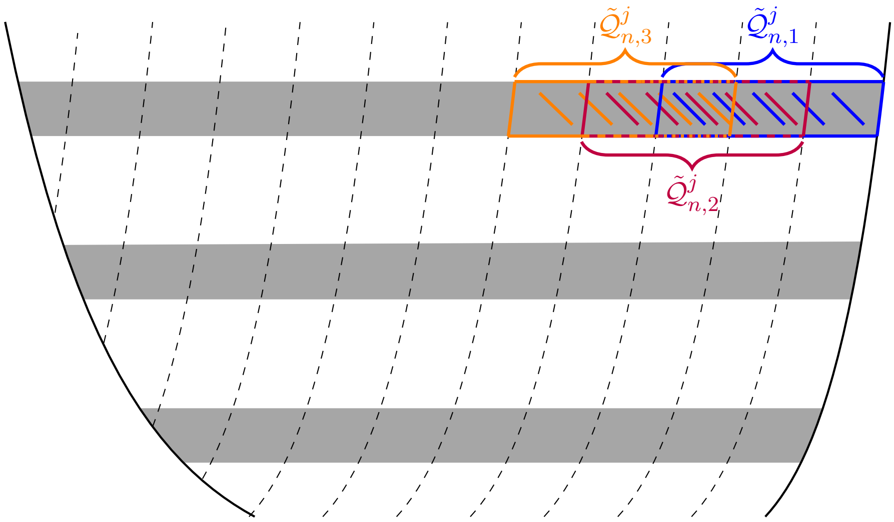
$$\beta_1 > \frac{72}{71} \cdot 15\pi. \tag{4.7.11}$$

Note that

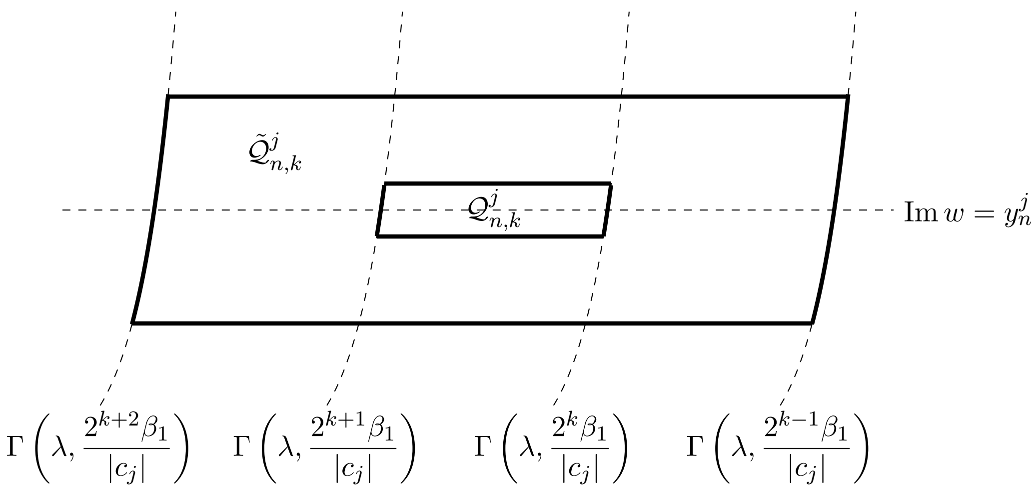
$$\frac{\theta_1}{3} < \frac{1}{18\pi} \arccos\left(\frac{5}{6}\right) < \frac{1}{18\pi} \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{72}. \tag{4.7.12}$$

For $w \in \mathcal{H}(\lambda-1, \alpha_1/|c_j|, \nu)\backslash\mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$, we have $|w| = (1+o(1))|\operatorname{Im} w|$ as $w \to \infty$. If $|\operatorname{Im} w - y^j_n| \le 5\pi\theta_1$, and ν and hence $|n|$ is sufficiently large, we deduce that

$$|n| \le |w| \le e^2|n|.$$



(a) (not to scale) The sets $\tilde{\mathcal{Q}}^j_{n,k}$ cover the grey strips, which are mapped into $\mathcal{H}(\lambda, 1/c^*, \nu)$ under h_j.



(b) The set $\tilde{\mathcal{Q}}^j_{n,k}$ in more detail and its subset $\mathcal{Q}^j_{n,k}$.

Figure 4.4: An illustration of the sets $\mathcal{Q}^j_{n,k}$ and $\tilde{\mathcal{Q}}^j_{n,k}$.

For $w \in \tilde{\mathcal{Q}}^j_{n,k}$, this implies that

$$\operatorname{Re} w \geq \lambda \log |w| - \log \frac{2^{k+2}\beta_1}{|c_j|} \geq \lambda \log |n| - 2|\lambda| - \log \frac{2^{k+2}\beta_1}{|c_j|} \tag{4.7.13}$$

and

$$\operatorname{Re} w \leq \lambda \log |w| - \log \frac{2^{k-1}\beta_1}{|c_j|} \leq \lambda \log |n| + 2|\lambda| - \log \frac{2^{k-1}\beta_1}{|c_j|}. \tag{4.7.14}$$

Here, the summands $\pm 2|\lambda|$ are only needed if $\lambda < 0$. Let $w \in \mathcal{Q}^j_{n,k} \subset \tilde{\mathcal{Q}}^j_{n,k}$. By Lemma

4.7.2 with $\varepsilon = \theta_1/3$, (4.7.13), the definition of θ_1 and (4.7.12), we have

$$\begin{aligned}\operatorname{Re} h_j(w) &\geq \operatorname{Re} w + \left(1 - \frac{\theta_1}{3}\right) 2^k \beta_1 \cos(\frac{4}{3}\theta_1) \\ &> \lambda \log|n| - 2|\lambda| - \log \frac{2^{k+2}\beta_1}{|c_j|} + \left(1 - \frac{\theta_1}{3}\right) 2^k \beta_1 \cdot \frac{5}{6} \\ &> \lambda \log|n| + \frac{9}{10} 2^k \beta_1 \cdot \frac{5}{6} = \lambda \log|n| + \frac{3}{4} 2^k \beta_1\end{aligned}$$

if β_1 is sufficiently large. Analogously,

$$\begin{aligned}\operatorname{Re} h_j(w) &\leq \operatorname{Re} w + \left(1 + \frac{\theta_1}{3}\right) 2^{k+1} \beta_1 \\ &\leq \lambda \log|n| + 2|\lambda| - \log \frac{2^{k-1}\beta_1}{|c_j|} + \left(1 + \frac{\theta_1}{3}\right) 2^{k+1} \beta_1 \\ &< \lambda \log|n| + \frac{5}{4} 2^{k+1} \beta_1 = \lambda \log|n| + \frac{5}{2} 2^k \beta_1\end{aligned}$$

if β_1 is sufficiently large. Moreover, by Lemma 4.7.2, (4.7.11) and (4.7.12),

$$\begin{aligned}|\operatorname{Im} h_j(w) - y_n^j| &\leq |\operatorname{Im} h_j(w) - \operatorname{Im} w| + |\operatorname{Im} w - y_n^j| \\ &\leq \left(1 + \frac{\theta_1}{3}\right) 2^{k+1} \beta_1 \sin\left(\frac{4}{3}\theta_1\right) + |\operatorname{Im} w - y_n^j| \\ &< \frac{17}{16} 2^{k+1} \beta_1 \frac{4}{3}\theta_1 + 2^k \frac{\beta_1}{6}\theta_1 = 3 \cdot 2^k \beta_1 \theta_1.\end{aligned}$$

Thus $h_j(\mathcal{Q}_{n,k}^j) \subset \mathcal{R}_{n,k}^j$.

In the following, we show that $h_j(\partial\tilde{\mathcal{Q}}_{n,k}^j) \cap \tilde{\mathcal{R}}_{n,k}^j = \emptyset$. Since we have already shown that $h_j(\tilde{\mathcal{Q}}_{n,k}^j) \cap \tilde{\mathcal{R}}_{n,k}^j \neq \emptyset$, this implies that $h_j(\tilde{\mathcal{Q}}_{n,k}^j) \supset \tilde{\mathcal{R}}_{n,k}^j$.

If $w \in \Gamma(\lambda, 2^{k-1}\beta_1/|c_j|)$ and β_1 is large, then by Lemma 4.7.2, (4.7.14) and (4.7.11),

$$\begin{aligned}\operatorname{Re} h_j(w) &\leq \operatorname{Re} w + \left(1 + \frac{\theta_1}{3}\right) 2^{k-1} \beta_1 \\ &\leq \lambda \log|n| + 2|\lambda| - \log \frac{2^{k-1}\beta_1}{|c_j|} + \left(1 + \frac{\theta_1}{3}\right) 2^{k-1} \beta_1 \\ &< \lambda \log|n| + \frac{5}{4} 2^{k-1} \beta_1 = \lambda \log|n| + \frac{5}{8} 2^k \beta_1.\end{aligned}$$

If $w \in \Gamma(\lambda, 2^{k+2}\beta_1/|c_j|)$ and $|\operatorname{Im} w - y_n^j| \leq 5\pi\theta_1$, then by Lemma 4.7.2, (4.7.13), the definition of θ_1 and (4.7.12), we have

$$\begin{aligned}\operatorname{Re} h_j(w) &\geq \operatorname{Re} w + \left(1 - \frac{\theta_1}{3}\right) 2^{k+2} \beta_1 \cos\left(5\pi\theta_1 + \frac{\theta_1}{3}\right) \\ &\geq \lambda \log|n| - 2|\lambda| - \log \frac{2^{k+2}\beta_1}{|c_j|} + \left(1 - \frac{\theta_1}{3}\right) 2^{k+2} \beta_1 \cos(6\pi\theta_1) \\ &> \lambda \log|n| - 2|\lambda| - \log \frac{2^{k+2}\beta_1}{|c_j|} + \left(1 - \frac{\theta_1}{3}\right) 2^{k+2} \beta_1 \cdot \frac{5}{6} \\ &> \lambda \log|n| + \frac{9}{10} \cdot 2^{k+2} \beta_1 \cdot \frac{5}{6} = \lambda \log|n| + 3 \cdot 2^k \beta_1,\end{aligned}$$

provided β_1 is sufficiently large.

If $|\operatorname{Im} w - y_n^j| = 5\pi\theta_1$ and $w \in \mathcal{H}(\lambda, 2^{k+2}\beta_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, 2^{k-1}\beta_1/|c_j|, \nu)$, then by Lemma 4.7.2, (4.7.12) and (4.7.11),

$$\begin{aligned}|\operatorname{Im} h_j(w) - y_n^j| &\geq |\operatorname{Im} h_j(w) - \operatorname{Im}(w)| - |\operatorname{Im} w - y_n^j| \\ &\geq \left(1 - \frac{\theta_1}{3}\right) 2^{k-1}\beta_1 \sin\left(5\pi\theta_1 - \frac{\theta_1}{3}\right) - 5\pi\theta_1 \\ &\geq \frac{71}{72} 2^k \beta_1 \left(5\theta_1 - \frac{\theta_1}{3\pi}\right) - 2^k 5\pi\theta_1\beta_1 \cdot \frac{71}{72 \cdot 15\pi} \\ &> \frac{71}{72} 2^k \beta_1 \left(5 - \frac{4}{9}\right)\theta_1 > 4 \cdot 2^k \beta_1 \theta_1.\end{aligned}$$

Thus $h_j(\partial\tilde{\mathcal{Q}}_{n,k}^j) \subset \mathbb{C} \setminus \tilde{\mathcal{R}}_{n,k}^j$. □

Next, we prove that the density of $q(\mathcal{F}_j)$ in $\tilde{Q}_{n,k}^j$ is bounded below by a positive constant.

Lemma 4.7.5. *There are $\alpha_1, \beta_1, \nu, \delta > 0$ such that for all $j \in \{1, ..., d\}$, all $n \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $\tilde{\mathcal{Q}}_{n,k}^j \subset \mathcal{H}(\lambda - 1, \alpha_1/|c_j|, \nu)$, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), \tilde{\mathcal{Q}}_{n,k}^j) \geq \delta.$$

Proof. Suppose that α_1, β_1, ν are chosen such that the conclusions of Lemmas 4.7.3 and 4.7.4 hold. By Section 4.5, in particular Lemma 4.5.5, the function $h_j = q \circ f \circ \varphi_j$ has no critical points in $\tilde{\mathcal{Q}}_{n,k}^j$ if ν and β_1 are sufficiently large. By Lemma 4.7.4, $h_j(\tilde{\mathcal{Q}}_{n,k}^j) \supset \tilde{\mathcal{R}}_{n,k}^j$ and $h_j(\mathcal{Q}_{n,k}^j) \subset \mathcal{R}_{n,k}^j$. Let U be the component of $h_j^{-1}(\tilde{\mathcal{R}}_{n,k}^j)$ containing $\mathcal{Q}_{n,k}^j$. Then $\mathcal{Q}_{n,k}^j \subset U \subset \tilde{\mathcal{Q}}_{n,k}^j$. Since $\tilde{\mathcal{R}}_{n,k}^j$ is simply connected, h_j maps U conformally onto $\tilde{\mathcal{R}}_{n,k}^j$. Let $\psi : \tilde{\mathcal{R}}_{n,k}^j \to U$ be the corresponding inverse function. By Lemma 4.7.3,

$$h_j(\tilde{\mathcal{Q}}_{n,k}^j) \subset \mathcal{H}\left(\lambda, \frac{1}{c^*}, \nu_0\right) \subset \mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup [0, \infty)) = q(\mathcal{S}_l)$$

for all $l \in \{1, ..., d\}$. Hence, there exists $l \in \{1, ..., d\}$ such that $(f \circ \varphi_j)(\tilde{\mathcal{Q}}_{n,k}^j) \subset \mathcal{S}_l$. We have $\psi(q(\mathcal{F}_l) \cap \tilde{\mathcal{R}}_{n,k}^j) = q(\mathcal{F}_j) \cap U$. By the Koebe distortion theorem, ψ has bounded distortion in $\mathcal{R}_{n,k}^j$ independent of n, k and j. We deduce that

$$\begin{aligned}\operatorname{dens}(q(\mathcal{F}_j), \tilde{\mathcal{Q}}_{n,k}^j) &\geq \operatorname{dens}(q(\mathcal{F}_j), \psi(\mathcal{R}_{n,k}^j)) \cdot \operatorname{dens}(\psi(\mathcal{R}_{n,k}^j), \tilde{\mathcal{Q}}_{n,k}^j) \\ &= \operatorname{dens}(\psi(q(\mathcal{F}_l) \cap \mathcal{R}_{n,k}^j), \psi(\mathcal{R}_{n,k}^j)) \cdot \operatorname{dens}(\psi(\mathcal{R}_{n,k}^j), \tilde{\mathcal{Q}}_{n,k}^j) \\ &\geq a \operatorname{dens}(q(\mathcal{F}_l), \mathcal{R}_{n,k}^j) \cdot \operatorname{dens}(\mathcal{Q}_{n,k}^j, \tilde{\mathcal{Q}}_{n,k}^j)\end{aligned}$$

for some constant $a > 0$ that does not depend on n, k and j. If β_1 is sufficiently large, then by Lemma 4.6.13,

$$\operatorname{dens}(q(\mathcal{F}_l), \mathcal{R}_{n,k}^j) \geq \eta_0.$$

Moreover, by Lemma 4.4.2,

$$\operatorname{meas}(\mathcal{Q}_{n,k}^j) \geq \frac{2}{3}\log 2 \cdot 2\theta_1 \quad \text{and} \quad \operatorname{meas}(\tilde{\mathcal{Q}}_{n,k}^j) \leq 2\log 8 \cdot 10\pi\theta_1.$$

Hence

$$\operatorname{dens}(q(\mathcal{F}_j), \tilde{\mathcal{Q}}_{n,k}^j) \geq a\eta_0 \frac{\log 2}{15\pi \log 8} =: \delta.$$ □

The last lemma of this section says that there is a positive lower bound for the density of $q(\mathcal{F}_j)$ in any sufficiently large rectangle contained in $\mathcal{H}(\lambda-1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$.

Lemma 4.7.6. *There are $\alpha_1, \beta_1, \nu, D_1, \eta_1 > 0$ such that for all $j \in \{1, ..., d\}$ and any rectangle $S \subset \mathcal{H}(\lambda-1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ with sides parallel to the real and imaginary axis and side lengths at least D_1, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), S) \geq \eta_1.$$

Proof. Suppose

$$D_1 \geq \max\{5 \log 8,\ 2\pi + 10\pi\theta_1\}.$$

Let $S \subset \mathcal{H}(\lambda-1, \alpha_1/|c_j|, \nu) \setminus \mathcal{H}(\lambda, \beta_1/|c_j|, \nu)$ be a rectangle with sides parallel to the real and imaginary axis and side lengths at least D_1. By the definition of $\tilde{\mathcal{Q}}^j_{n,k}$ and Lemma 4.4.2, there are $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that

$$\tilde{\mathcal{Q}}^j_{n,k} \subset S.$$

If, in addition, the side lengths of S do not exceed $2D_1$, then by Lemma 4.7.5 and Lemma 4.4.2,

$$\operatorname{dens}(q(\mathcal{F}_j), S) \geq \operatorname{dens}(q(\mathcal{F}_j), \tilde{\mathcal{Q}}^j_{n,k}) \cdot \operatorname{dens}(\tilde{\mathcal{Q}}^j_{n,k}, S) \geq \delta \frac{2/3 \log 8 \cdot 10\pi\theta_1}{4D_1^2}.$$

Since any general rectangle with side lengths at least D_1 can be written as the union of rectangles with side lengths between D_1 and $2D_1$ which are disjoint up to the boundary, the claim follows. $\square$

4.8 The set $q(\mathcal{F}(f))$: third part

For $\nu > 0$, let

$$\mathcal{G}_\nu := \{w \in \mathbb{C} :\ |\operatorname{Im} w| \geq \nu\}.$$

In this section we investigate the density of $q(\mathcal{F}(f))$ in subsets of $\mathcal{G}_\nu \setminus \mathcal{H}(\lambda-1, \beta_2/|c_j|, \nu)$, for large $\beta_2 > 0$. The first lemma gives an approximation for h_j in $\mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$.

Lemma 4.8.1. *Let $\varepsilon > 0$ and $j \in \{1, ..., d\}$. There are $\beta_2, \nu > 0$ such that for all $w \in \mathcal{G}_\nu \setminus \mathcal{H}(\lambda-1, \beta_2/|c_j|, \nu)$, we have*

$$\left| \frac{h_j(w)}{(-c_j/d)^d e^{-dw} w^{-m}} - 1 \right| < \varepsilon.$$

Proof. By Corollary 4.3.4,

$$\begin{aligned} f(\varphi_j(w)) &= \varphi_j(w) - \frac{1}{q'(\varphi_j(w))}(1 + o(1)) - \frac{c_j e^{-w}}{p(\varphi_j(w))} \\ &= O(|w|^{1/d}) - \frac{c_j}{d} e^{-w} \varphi_j(w)^{-m}(1 + o(1)) \end{aligned} \tag{4.8.1}$$

as $w \to \infty$. Note that the $O(\cdot)$ and $o(\cdot)$-terms do not depend on β_2. For $w \in \mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$, we have

$$\left| \frac{c_j}{d} e^{-w} \varphi_j(w)^{-m} \right| = \left| \frac{c_j}{d} e^{-w} \right| \cdot |w|^{\lambda - 1 + 1/d}(1 + o(1)) \geq \frac{\beta_2 |w|^{1/d}}{2d} \tag{4.8.2}$$

if $|w|$ is sufficiently large. In particular,

$$|f(\varphi_j(w))| \geq \frac{\beta_2}{4d}|w|^{1/d}$$

if β_2 and $|w|$ are sufficiently large, and hence

$$h_j(w) = q(f(\varphi_j(w))) = f(\varphi_j(w))^d(1 + o(1))$$

as $w \to \infty$ in $\mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$. Also, by (4.8.1) and (4.8.2),

$$\begin{aligned} \left|\frac{f(\varphi_j(w))}{(-c_j/d)e^{-w}\varphi_j(w)^{-m}} - 1\right| &= \left|\frac{O(|w|^{1/d})}{(-c_j/d)e^{-w}\varphi_j(w)^{-m}} + o(1)\right| \\ &\leq \frac{2d}{\beta_2}O(1) + o(1), \end{aligned}$$

where the $O(\cdot)$ and $o(\cdot)$-terms do not depend on β_2. Hence, we can achieve that

$$\left|\frac{f(\varphi_j(w))^d}{((-c_j/d)e^{-w}\varphi_j(w)^{-m})^d} - 1\right| \leq \frac{\varepsilon}{2}$$

by taking β_2 and ν sufficiently large. Also,

$$\left(-\frac{c_j}{d}e^{-w}\varphi_j(w)^{-m}\right)^d = \left(-\frac{c_j}{d}\right)^d e^{-dw}w^{-m}(1 + o(1))$$

as $w \to \infty$, whence the claim follows. □

We proceed similarly as in Section 4.7. That is, we begin by proving that h_j maps certain subsets of $\mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$ into $\mathcal{H}(\lambda, 1/c^*, \nu_0)$, and then apply the results of Section 4.6 to show that the density of $q(\mathcal{F}(f))$ in large bounded subsets of $\mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$ is bounded away from zero.

For $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $j \in \{1, ..., d\}$, let $\mathcal{P}^j_{n,k}$ be the set of all

$$w \in \mathcal{H}\left(\lambda - 1, \frac{2^{k+2}\beta_2}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda - 1, \frac{2^{k-1}\beta_2}{|c_j|}, \nu\right)$$

satisfying

$$\frac{(2n-1)\pi}{d} \leq \operatorname{Im} w \leq \frac{2(n+1)\pi}{d}.$$

There are $\theta^j_{n,k} \in [-\pi, \pi)$ and $r^j_{n,k} > (2|n| - 2)\pi/d$ such that for all $w \in \mathcal{P}^j_{n,k}$, we have

$$|w| = r^j_{n,k}(1 + o(1)) \quad \text{and} \quad \arg(w) = \theta^j_{n,k} + o(1)$$

as $|n| \to \infty$. Let $t^j_{n,k} \in [2n\pi/d, 2(n+1)\pi/d)$ with

$$t^j_{n,k} \equiv \arg(-c_j) - \frac{m}{d}\theta^j_{n,k} \mod \frac{2\pi}{d}.$$

Lemma 4.8.2. *Fix $\theta^* \in (0, \pi/(4d))$. There are $\beta_2, \nu > 0$ such that the following holds.*

Let $j \in \{1, ..., d\}$, $k \in \mathbb{N}$ and $w \in \mathcal{H}(\lambda-1, 2^{k+2}\beta_2/|c_j|, \nu) \setminus \mathcal{H}(\lambda-1, 2^{k-1}\beta_2/|c_j|, \nu)$ so that there exists $n \in \mathbb{Z}$ with $t^j_{n,k} - \pi/(4d) \leq \operatorname{Im} w \leq t^j_{n,k} - \theta^*$. Let $\beta \in [2^{k-1}\beta_2, 2^{k+2}\beta_2]$ so that $w \in \Gamma(\lambda - 1, \beta/|c_j|)$, and set $\theta := t^j_{n,k} - \operatorname{Im} w$. Then

$$\frac{3}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k}\cos\left(\frac{4d}{3}\theta\right) < \operatorname{Re} h_j(w) < \frac{5}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k}$$

and

$$\frac{1}{\pi}\left(\frac{\beta}{d}\right)^d r^j_{n,k}d\theta < \operatorname{Im} h_j(w) < \frac{5}{3}\left(\frac{\beta}{d}\right)^d r^j_{n,k}d\theta.$$

Proof. Let β_2, ν be chosen such that the conclusion of Lemma 4.8.1 holds for $\varepsilon = d\theta^*/(2\pi)$. Then

$$\left|\frac{h_j(w)}{(-c_j/d)^d e^{-dw} w^{-m}} - 1\right| < \frac{d\theta^*}{2\pi} < \frac{1}{8}.$$

Thus

$$\frac{7}{8}\left(\frac{|c_j|}{d}\right)^d e^{-d\operatorname{Re} w}|w|^{-m} \leq |h_j(w)| \leq \frac{9}{8}\left(\frac{|c_j|}{d}\right)^d e^{-d\operatorname{Re} w}|w|^{-m}. \tag{4.8.3}$$

Since $w \in \Gamma(\lambda - 1, \beta/|c_j|)$, we have

$$|w|^{-1-m} e^{-d\operatorname{Re} w} = |w|^{d(\lambda-1)} e^{-d\operatorname{Re} w} = \left(\frac{\beta}{|c_j|}\right)^d.$$

Hence

$$\left(\frac{|c_j|}{d}\right)^d e^{-d\operatorname{Re} w}|w|^{-m} = \left(\frac{\beta}{d}\right)^d |w| = \left(\frac{\beta}{d}\right)^d r^j_{n,k}(1 + o(1))$$

as $|n| \to \infty$. Inserting the last equation into (4.8.3) yields

$$\frac{3}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k} < |h_j(w)| < \frac{5}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k} \tag{4.8.4}$$

if $|n|$ is sufficiently large. Also, by Lemma 4.8.1 and the inequality $\arcsin(x) \leq (\pi/2)x$ for $x \in [0, 1]$, we have

$$\left|\arg(h_j(w)) - \arg\left(\left(-\frac{c_j}{d}\right)^d e^{-dw} w^{-m}\right)\right| < \arcsin\left(\frac{d\theta^*}{2\pi}\right) \leq \frac{d\theta^*}{4}. \tag{4.8.5}$$

We have

$$\begin{aligned}
\arg\left(\left(-\frac{c_j}{d}\right)^d e^{-dw} w^{-m}\right) &\equiv d\arg(-c_j) - d\operatorname{Im} w - m\arg w \\
&\equiv d\arg(-c_j) - dt^j_{n,k} + d\theta - m\theta^j_{n,k} + o(1) \\
&\equiv d\theta + o(1) \mod 2\pi
\end{aligned}$$

as $|n| \to \infty$. By (4.8.5) and since $\theta \geq \theta^*$, this yields

$$\frac{2d\theta}{3} < \arg(h_j(w)) < \frac{4d}{3}\theta. \tag{4.8.6}$$

From (4.8.4), (4.8.6) and the fact that $(2/\pi)x \leq \sin x \leq x$ for $x \in [0, \pi/2]$, we deduce

$$\begin{aligned}
&\operatorname{Re} h_j(w) \leq |h_j(w)| < \frac{5}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k}, \\
&\operatorname{Re} h_j(w) = |h_j(w)|\cos(\arg(h_j(w))) > \frac{3}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k}\cos\left(\frac{4d}{3}\theta\right), \\
&\operatorname{Im} h_j(w) = |h_j(w)|\sin(\arg(h_j(w))) < \frac{5}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k}\sin\left(\frac{4d}{3}\theta\right) \leq \frac{5}{3}\left(\frac{\beta}{d}\right)^d r^j_{n,k} d\theta, \\
&\operatorname{Im} h_j(w) = |h_j(w)|\sin(\arg(h_j(w))) > \frac{3}{4}\left(\frac{\beta}{d}\right)^d r^j_{n,k}\sin\left(\frac{2d\theta}{3}\right) \geq \frac{1}{\pi}\left(\frac{\beta}{d}\right)^d r^j_{n,k} d\theta. \ \square
\end{aligned}$$

Next, we define several sets, starting with subsets $\mathcal{T}^j_{n,k}, \tilde{\mathcal{T}}^j_{n,k} \subset \mathcal{G}_\nu \backslash \mathcal{H}(\lambda-1, \beta_2/|c_j|, \nu)$. Fix

$$\theta_2 \in \left(0, \frac{3}{2 \cdot 4^{d+1} d\pi} \arccos\left(\frac{11}{12}\right)\right).$$

For $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $j \in \{1, ..., d\}$, let $\mathcal{T}^j_{n,k}$ be the set of all

$$w \in \mathcal{H}\left(\lambda - 1, \frac{2^{k+1}\beta_2}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda - 1, \frac{2^k\beta_2}{|c_j|}, \nu\right)$$

satisfying

$$t^j_{n,k} - \theta_2 \leq \operatorname{Im} w \leq t^j_{n,k} - \frac{\theta_2}{2}.$$

Also, let $\tilde{\mathcal{T}}^j_{n,k}$ be the set of all

$$w \in \mathcal{H}\left(\lambda - 1, \frac{2^{k+2}\beta_2}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda - 1, \frac{2^{k-1}\beta_2}{|c_j|}, \nu\right)$$

satisfying

$$t^j_{n,k} - 2 \cdot 4^d \pi \theta_2 \leq \operatorname{Im} w \leq t^j_{n,k} - \frac{1}{5 \cdot 4^d \pi}\theta_2.$$

Note that $\mathcal{T}^j_{n,k} \subset \tilde{\mathcal{T}}^j_{n,k}$. See Figure 4.5 for an illustration of these sets.

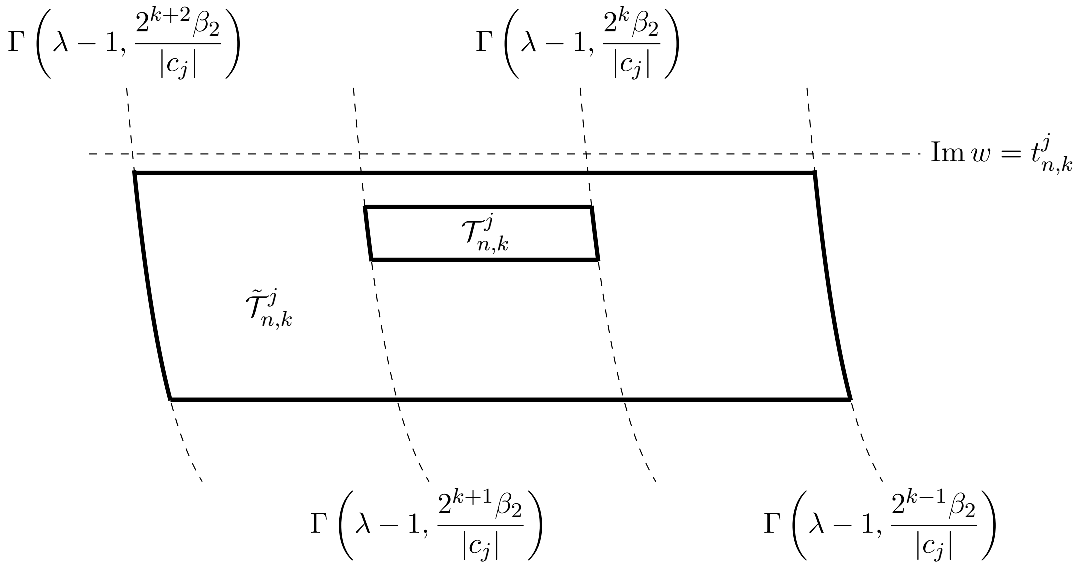


Figure 4.5: An illustration of the sets $\mathcal{T}^j_{n,k}$ and $\tilde{\mathcal{T}}^j_{n,k}$.

Moreover, define rectangles $\mathcal{U}^j_{n,k}$ and $\tilde{\mathcal{U}}^j_{n,k}$ as follows. Let $\mathcal{U}^j_{n,k}$ be the set of all $v \in \mathbb{C}$ satisfying

$$\frac{11}{16}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} < \operatorname{Re} v < \frac{5}{4}\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k}$$

and

$$\frac{1}{\pi}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} d\theta_2 < \operatorname{Im} v < \frac{5}{3}\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k} d\theta_2.$$

Also, let $\tilde{\mathcal{U}}^j_{n,k}$ be the rectangle containing all $v \in \mathbb{C}$ satisfying

$$\frac{5}{8}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} < \operatorname{Re} v < \frac{11}{8}\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k}$$

and

$$\frac{1}{3\pi}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} d\theta_2 < \operatorname{Im} v < 2\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k} d\theta_2.$$

Note that $\mathcal{U}^j_{n,k} \subset \tilde{\mathcal{U}}^j_{n,k}$. Recall that $c^* = \max_l |c_l|$.

Lemma 4.8.3. *There are $\beta_2, \nu > 0$ such that for all $n \in \mathbb{Z}$ with $(2|n|-2)\pi/d \geq \nu$, all $k \in \mathbb{N}$ and all $j \in \{1, ..., d\}$, we have*

$$\tilde{\mathcal{U}}^j_{n,k} \subset \mathcal{H}\left(\lambda, \frac{1}{c^*}, \nu\right).$$

Proof. Suppose $\beta_2 \geq 3\pi/(2\theta_2)$. Note that this implies $\beta_2/d > 1$. Let $v \in \tilde{\mathcal{U}}^j_{n,k}$. Since $r^j_{n,k} > (2|n|-2)\pi/d \geq \nu$, we have

$$\operatorname{Im} v > \frac{1}{3\pi}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} d\theta_2 > \frac{1}{3\pi}\left(\frac{2^k\beta_2}{d}\right)^d \nu d\theta_2 > \frac{2\beta_2}{3\pi d}\nu d\theta_2 \geq \nu.$$

Also,

$$\operatorname{Im} v < 2\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k} d\theta_2 = \frac{16}{5}2^d d\theta_2 \cdot \frac{5}{8}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} < \frac{16}{5}2^d d\theta_2 \operatorname{Re} v,$$

and hence

$$|v| \leq |\operatorname{Re} v| + |\operatorname{Im} v| < \left(1 + \frac{16}{5}2^d d\theta_2\right)\operatorname{Re} v.$$

Thus

$$\operatorname{Re} v \geq \frac{1}{1+(16/5)2^d d\theta_2}|v| \geq \lambda \log|v| - \log\frac{1}{c^*}$$

if ν and hence $r^j_{n,k}$ and $|v|$ are sufficiently large. □

Recall that $\mathcal{G}_\nu = \{w \in \mathbb{C} : |\operatorname{Im} w| \geq \nu\}$.

Lemma 4.8.4. *There are $\beta_2, \nu > 0$ such that for all $j \in \{1, ..., d\}$, all $k \in \mathbb{N}$ and all $n \in \mathbb{Z}$ with $\tilde{\mathcal{T}}^j_{n,k} \subset \mathcal{G}_\nu$, we have*

$$h_j(\mathcal{T}^j_{n,k}) \subset \mathcal{U}^j_{n,k} \quad \text{and } h_j(\tilde{\mathcal{T}}^j_{n,k}) \supset \tilde{\mathcal{U}}^j_{n,k}.$$

Proof. First suppose that $w \in \mathcal{T}^j_{n,k}$. Then by Lemma 4.8.2 and the fact that $\theta_2 < 3/(2\cdot 4^{d+1}d\pi)\arccos(11/12) < 3/(4d)\arccos(11/12)$, we have

$$\begin{aligned}
\operatorname{Re} h_j(w) &> \frac{3}{4}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k}\cos\left(\frac{4d}{3}\theta_2\right) > \frac{11}{16}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k},\\
\operatorname{Re} h_j(w) &< \frac{5}{4}\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k},\\
\operatorname{Im} h_j(w) &> \frac{1}{\pi}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} d\theta_2,\\
\operatorname{Im} h_j(w) &< \frac{5}{3}\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k} d\theta_2.
\end{aligned}$$

Hence, $h_j(\mathcal{T}^j_{n,k}) \subset \mathcal{U}^j_{n,k}$.

Let us now show that $h_j(\partial \tilde{T}^j_{n,k}) \cap \tilde{\mathcal{U}}^j_{n,k} = \emptyset$. Lemma 4.8.2 yields the following. For $w \in \Gamma(\lambda - 1, 2^{k-1}\beta_2/|c_j|)$ with $t^j_{n,k} - 2 \cdot 4^d \pi\theta_2 \leq \operatorname{Im} w \leq t^j_{n,k} - \theta_2/(5 \cdot 4^d \pi)$, we have

$$\operatorname{Re} h_j(w) < \frac{5}{4}\left(\frac{2^{k-1}\beta_2}{d}\right)^d r^j_{n,k} \leq \frac{5}{8}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k}.$$

If $w \in \Gamma(\lambda - 1, 2^{k+2}\beta_2/|c_j|)$ with $t^j_{n,k} - 2 \cdot 4^d \pi\theta_2 \leq \operatorname{Im} w \leq t^j_{n,k} - \theta_2/(5 \cdot 4^d \pi)$, then from the fact that $\theta_2 < 3/(2 \cdot 4^{d+1} d\pi) \arccos(11/12)$, we deduce

$$\operatorname{Re} h_j(w) > \frac{3}{4}\left(\frac{2^{k+2}\beta_2}{d}\right)^d r^j_{n,k} \cos\left(\frac{2d}{3} \cdot 4^{d+1}\pi\theta_2\right) > \frac{11}{8}\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k}.$$

For $w \in \mathcal{H}(\lambda - 1, 2^{k+2}\beta_2/|c_j|, \nu) \setminus \mathcal{H}(\lambda - 1, 2^{k-1}\beta_2/|c_j|, \nu)$ with $\operatorname{Im} w = t^j_{n,k} - 2 \cdot 4^d \pi\theta_2$, we have

$$\operatorname{Im} h_j(w) > 2\left(\frac{2^{k+1}\beta_2}{d}\right)^d r^j_{n,k} d\theta_2.$$

If $w \in \mathcal{H}(\lambda - 1, 2^{k+2}\beta_2/|c_j|, \nu) \setminus \mathcal{H}(\lambda - 1, 2^{k-1}\beta_2/|c_j|, \nu)$ and $\operatorname{Im} w = t^j_{n,k} - \theta_2/(5 \cdot 4^d \pi)$, then

$$\operatorname{Im} h_j(w) < \frac{1}{3\pi}\left(\frac{2^k\beta_2}{d}\right)^d r^j_{n,k} d\theta_2.$$

Thus $h_j(\partial \tilde{\mathcal{T}}^j_{n,k}) \cap \tilde{\mathcal{U}}^j_{n,k} = \emptyset$. Since $\mathcal{T}^j_{n,k} \subset \tilde{\mathcal{T}}^j_{n,k}$ and $h_j(\mathcal{T}^j_{n,k}) \subset \mathcal{U}^j_{n,k} \subset \tilde{\mathcal{U}}^j_{n,k}$, we deduce that $h_j(\tilde{\mathcal{T}}^j_{n,k}) \supset \tilde{\mathcal{U}}^j_{n,k}$. □

Next, we show that the density of $q(\mathcal{F}_j)$ in $\tilde{\mathcal{T}}^j_{n,k}$ is bounded below by a positive constant.

Lemma 4.8.5. *There are $\delta > 0$ and $\nu > 0$ such that for all $j \in \{1, ..., d\}$, all $k \in \mathbb{N}$ and all $n \in \mathbb{Z}$ such that $\tilde{\mathcal{T}}^j_{n,k} \subset \mathcal{G}_\nu$, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), \tilde{\mathcal{T}}^j_{n,k}) \geq \delta.$$

Proof. We only sketch the proof, since it is similar to the one of Lemma 4.7.5. By Lemma 4.8.3,

$$\tilde{\mathcal{U}}^j_{n,k} \subset \mathcal{H}\left(\lambda, \frac{1}{c^*}, \nu\right).$$

By Lemma 4.8.4, $h_j(\tilde{\mathcal{T}}^j_{n,k}) \supset \tilde{\mathcal{U}}^j_{n,k}$ and $h_j(\mathcal{T}^j_{n,k}) \subset \mathcal{U}^j_{n,k}$. Let $V \subset \tilde{\mathcal{T}}^j_{n,k}$ be the component of $h_j^{-1}(\tilde{\mathcal{U}}^j_{n,k})$ containing $\mathcal{T}^j_{n,k}$. As in the proof of Lemma 4.7.5, we get that $f(\varphi_j(V)) \subset \mathcal{S}_l$ for some $l \in \{1, ..., d\}$, and that

$$\operatorname{dens}(q(\mathcal{F}_j), \tilde{\mathcal{T}}^j_{n,k}) \geq a \operatorname{dens}(q(\mathcal{F}_l), \mathcal{U}^j_{n,k}) \cdot \operatorname{dens}(\mathcal{T}^j_{n,k}, \tilde{\mathcal{T}}^j_{n,k})$$

for some constant $a > 0$ that does not depend on n, k or j. If ν and hence $r^j_{n,k}$ is sufficiently large, then Lemma 4.6.13 yields

$$\operatorname{dens}(q(\mathcal{F}_l), \mathcal{U}^j_{n,k}) \geq \eta_0.$$

Also, the density of $\mathcal{T}^j_{n,k}$ in $\tilde{\mathcal{T}}^j_{n,k}$ is bounded below independent of n, k and j, whence the claim follows. □

The final result of this section says that the density of $q(\mathcal{F}_j)$ in large rectangles in $\mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$ is bounded below by a positive constant.

Lemma 4.8.6. *There are $\beta_2, \nu, D_2, \eta_2 > 0$ such that for all $j \in \{1, ..., d\}$ and any rectangle $S \subset \mathcal{G}_\nu \setminus \mathcal{H}(\lambda - 1, \beta_2/|c_j|, \nu)$ with sides parallel to the real and imaginary axis and side lengths at least D_2, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), S) \geq \eta_2.$$

Proof. This is proved the same way as Lemma 4.7.6, using Lemma 4.8.5. □

4.9 The set $q(\mathcal{F}(f))$: conclusions

In this section we combine the results of Sections 4.6-4.8 to show that the density of $q(\mathcal{F}_j)$ in large bounded subsets of the complex plane is bounded away from zero.

Lemma 4.9.1. *There are $D, \eta_3 > 0$ such that for all $j \in \{1, ..., d\}$ and any square $S \subset \mathbb{C}$ with sides parallel to the real and imaginary axis and side lengths at least D, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), S) \geq \eta_3.$$

Proof. Set

$$\mathcal{E}_1 := \mathcal{H}\left(\lambda, \frac{\beta_1}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda, \frac{1}{|c_j|}, \nu\right)$$

and

$$\mathcal{E}_2 := \mathcal{H}\left(\lambda - 1, \frac{\beta_2}{|c_j|}, \nu\right) \setminus \mathcal{H}\left(\lambda - 1, \frac{\alpha_1}{|c_j|}, \nu\right).$$

Let γ_1 and γ_2 denote the left boundary curves of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, parametrised by $y = \operatorname{Im} z$. Justified by Lemma 4.4.2, assume that ν is so large that

$$|\gamma_k'(y)| < \frac{1}{10} \text{ for } |y| \geq \nu \text{ and } k \in \{1, 2\}. \tag{4.9.1}$$

Using the notation of Lemmas 4.6.13, 4.7.6 and 4.8.6, suppose

$$D > 2\nu + 5 \max\{D_0, D_1, D_2\} \tag{4.9.2}$$

and

$$D > 20 \max\left\{\log \beta_1, \log \frac{\beta_2}{\alpha_1}\right\}. \tag{4.9.3}$$

For $A \subset \mathbb{C}$, define

$$\operatorname{diam}_x(A) := \sup\{|\operatorname{Re}(z - w)| : z, w \in A\}$$

and

$$\operatorname{diam}_y(A) := \sup\{|\operatorname{Im}(z - w)| : z, w \in A\}.$$

Set

$$S_+ := S \cap \{z \in \mathbb{C} : \operatorname{Im} z \geq \nu\}, \quad S_- := S \cap \{z \in \mathbb{C} : \operatorname{Im} z \leq -\nu\},$$

and let

$$S_1 := \begin{cases} S_+ & \text{if } \operatorname{diam}_y(S_+) \geq \operatorname{diam}_y(S_-) \\ S_- & \text{otherwise.} \end{cases}$$

By (4.9.2), $\operatorname{diam}_y(S_1) > \max\{D_0, D_1, D_2\}$.

We divide S_1 into 5 rectangles $S_{1,1}, ..., S_{1,5}$ with $\operatorname{diam}_y(S_{1,k}) = \operatorname{diam}_y(S_1)$ and $\operatorname{diam}_x(S_{1,k}) = (1/5) \operatorname{diam}_x(S_1)$ for all $k \in \{1, ..., 5\}$ (see Figure 4.6).

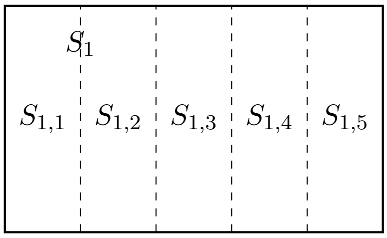


Figure 4.6: The dashed lines divide the rectangle S_1 (bounded by the solid lines) into five smaller rectangles.

By (4.9.2), $\operatorname{diam}_x(S_{1,k}) > \max\{D_0, D_1, D_2\}$. From (4.9.1), Lemma 4.4.2, (4.9.3) and the fact that S is a square of side length at least D, we deduce that

$$\begin{aligned}\operatorname{diam}_x(\mathcal{E}_l \cap S) &< \frac{1}{10}\operatorname{diam}_y(S) + 2\max\left\{\log\beta_1, \log\frac{\beta_2}{\alpha_1}\right\}\\ &< \frac{1}{10}\operatorname{diam}_y(S) + \frac{1}{10}D \le \frac{1}{5}\operatorname{diam}_x(S)\end{aligned}$$

for $l \in \{1,2\}$. Thus $\mathcal{E}_1$ and $\mathcal{E}_2$ each intersect at most two of the rectangles $S_{1,k}$. Hence there exists $l \in \{1, ..., 5\}$ such that $S_{1,l}$ does not intersect $\mathcal{E}_1 \cup \mathcal{E}_2$. This implies that $S_{1,l}$ satisfies the assumptions of one of Lemmas 4.6.13, 4.7.6 and 4.8.6. Thus

$$\operatorname{dens}(q(\mathcal{F}_j), S_{1,l}) \ge \min\{\eta_0, \eta_1, \eta_2\}$$

and

$$\begin{aligned}\operatorname{dens}(q(\mathcal{F}_j), S) &\ge \operatorname{dens}(q(\mathcal{F}_j), S_{1,l}) \cdot \operatorname{dens}(S_{1,l}, S)\\ &\ge \min\{\eta_0, \eta_1, \eta_2\} \cdot \frac{1}{10}\frac{\operatorname{diam}_x(S)(\operatorname{diam}_y(S) - 2\nu)}{(\operatorname{diam}_x S)^2}\\ &\ge \min\{\eta_0, \eta_1, \eta_2\} \cdot \frac{1}{10}\left(1 - \frac{2\nu}{D}\right).\end{aligned}$$

□

The following corollary is an immediate consequence of Lemma 4.9.1.

Corollary 4.9.2. *There are $r_1, \eta > 0$ such that for all $z \in \mathbb{C}$, all $r \ge r_1$ and all $j \in \{1, ..., d\}$, we have*

$$\operatorname{dens}(q(\mathcal{F}_j), \mathcal{D}(z, r)) \ge \eta.$$

Remark 4.9.3. Corollary 4.9.2 says that $\mathbb{C} \setminus q(\mathcal{F}_j)$ is thin at infinity.

4.10 Proof of Theorem B

Proof of Theorem B. We verify the assumptions of Theorem A. To this end, we show that there exists $R_1 > 0$ such that $\mathcal{J}(f)$ is uniformly thin at $\mathcal{P}(f) \setminus \overline{\mathcal{D}(0, R_1)}$, and that $\mathcal{J}(f)$ is thin at infinity. Then the assumptions of Theorem A are satisfied for $\mathcal{P}_1 := \mathcal{P}(f) \cap \overline{\mathcal{D}(0, R_1)}$, because $\mathcal{P}(f) \cap \mathcal{J}(f)$ is a finite set by Lemma 4.5.2.

Let $r_2 > 0$ such that

(a) $|q'(z)| \ge (d/2)|z|^{d-1}$ for all $z \in \mathbb{C}$ with $|z| \ge r_2$;

(b) each $z_0 \in \mathcal{P}(f)$ with $|z_0| \ge r_2$ is a simple zero of g and hence a superattracting fixed point of f. Justified by Corollary 4.5.6, we also assume that there exists $j \in \{1, ..., d\}$ with $z_0 \in \mathcal{S}_j$ and $\operatorname{dist}(z_0, \partial\mathcal{S}_j) \ge 3$. Moreover, suppose that $r_2 \ge r_0$, for r_0 as in Lemma 4.6.5.

Let r_1 be as in Corollary 4.9.2. First, we show that there exists $\eta_4 > 0$ such that for all $j \in \{1, ..., d\}$, all $z \in \mathcal{S}_j$ with $|z| \geq r_2$ and all $r \geq 8r_1/(d|z|^{d-1})$ with $\mathcal{D}(z, 2r) \subset \mathcal{S}_j$, we have

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z, r)) \geq \eta_4. \tag{4.10.1}$$

Recall that q is injective in $\mathcal{S}_j$. By Koebe's theorems,

$$\mathcal{D}\left(q(z), \frac{1}{4}|q'(z)|r\right) \subset q(\mathcal{D}(z, r)) \subset \mathcal{D}(q(z), 4|q'(z)|r).$$

By (a) and the assumption on r, we have $(1/4)|q'(z)|r \geq r_1$. Hence, by Corollary 4.9.2,

$$\operatorname{dens}\left(q(\mathcal{F}_j), \mathcal{D}\left(q(z), \frac{1}{4}|q'(z)|r\right)\right) \geq \eta.$$

Thus

$$\begin{aligned} &\operatorname{dens}(q(\mathcal{F}_j), q(\mathcal{D}(z, r))) \\ &\geq \operatorname{dens}\left(\mathcal{D}\left(q(z), \frac{1}{4}|q'(z)|r\right), q(\mathcal{D}(z, r))\right) \cdot \operatorname{dens}\left(q(\mathcal{F}_j), \mathcal{D}\left(q(z), \frac{1}{4}|q'(z)|r\right)\right) \\ &\geq \operatorname{dens}\left(\mathcal{D}\left(q(z), \frac{1}{4}|q'(z)|r\right), \mathcal{D}(q(z), 4|q'(z)|r)\right) \cdot \eta = \frac{\eta}{256}. \end{aligned}$$

By the Koebe distortion theorem,

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z, r)) \geq \left(\frac{\min_{|\zeta - z| \leq r} |q'(\zeta)|}{\max_{|\zeta - z| \leq r} |q'(\zeta)|}\right)^2 \operatorname{dens}(q(\mathcal{F}_j), q(\mathcal{D}(z, r))) \geq \frac{\eta}{3^8 \cdot 256}.$$

This implies (4.10.1) with $\eta_4 = \eta/(3^8 \cdot 256)$.

Let us now prove that there exists $R_1 > 0$ such that $\mathcal{J}(f)$ is uniformly thin at $\mathcal{P}(f) \setminus \overline{\mathcal{D}(0, R_1)}$. Let $\delta_1 \in (0, 1)$, $z_0 \in \mathcal{P}(f)$ with $|z_0| > r_2 + 1$, and $z \in \mathcal{D}(z_0, \delta_1)$. By (b), $\mathcal{D}(z, 2\delta_1) \subset \mathcal{S}_j$. Also, $|z| \geq r_2$. If $|z - z_0| \geq 8r_1/(d|z|^{d-1})$, then by (4.10.1),

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z, |z - z_0|)) \geq \eta_4.$$

Now suppose that

$$|z - z_0| < \frac{8r_1}{d|z|^{d-1}}. \tag{4.10.2}$$

By Lemma 4.6.5, we have $\mathcal{D}(z_0, 1/(3d|z_0|^{d-1})) \subset \mathcal{F}(f)$. Hence

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z, |z - z_0|)) \geq \operatorname{dens}\left(\mathcal{D}\left(z_0, \frac{1}{3d|z_0|^{d-1}}\right), \mathcal{D}(z, |z - z_0|)\right).$$

The expression on the right hand side is bounded below independent of z_0 and z, provided (4.10.2) is satisfied. So $\mathcal{J}(f)$ is uniformly thin at $\mathcal{P}(f) \setminus \overline{\mathcal{D}(0, r_1 + 1)}$.

It remains to prove that $\mathcal{J}(f)$ is thin at infinity. Let R be as in Section 4.2 and let $r_3 > \max\{2R^{1/d}, r_2\}$. If r_3 is sufficiently large, then Lemma 4.2.2 yields that $\bigcup_{j=1}^d \partial\mathcal{S}_j \setminus \mathcal{D}(0, r_3)$ is contained in d pairwise disjoint halfstrips $T_1, ..., T_d$ of width one. We can assume that r_3 is so large that $\operatorname{dist}(T_k, T_l) \geq 1$ for $k \neq l$. Then for $|z| \geq r_3 + 3$, the set $\mathcal{D}(z, 3) \setminus \bigcup_{j=1}^d T_j$ contains a disk D of radius $1/2$. There is $j \in \{1, ..., d\}$ with $D \subset \mathcal{S}_j$. Let D' be the disk with the same centre as D and radius $1/4$. By (4.10.1), we have $\operatorname{dens}(\mathcal{F}(f), D') \geq \eta_4$, and hence

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z, 3)) \geq \operatorname{dens}(\mathcal{F}(f), D') \cdot \operatorname{dens}(D', \mathcal{D}(z, 3)) \geq \frac{\eta_4}{144}.$$

Let us now consider the case $|z| < r_3 + 3$. Let $\zeta_1, ..., \zeta_n \in \mathcal{D}(0, r_3 + 3)$ such that

$$\mathcal{D}(0, r_3 + 3) \subset \bigcup_{k=1}^{n} \mathcal{D}(\zeta_k, 1).$$

Then

$$\eta_5 := \min_{1 \leq k \leq n} \operatorname{dens}(\mathcal{F}(f), \mathcal{D}(\zeta_k, 1)) > 0.$$

For $z \in \mathcal{D}(0, r_3 + 3)$, let $k \in \{1, ..., n\}$ such that $z \in \mathcal{D}(\zeta_k, 1)$. Then $\mathcal{D}(\zeta_k, 1) \subset \mathcal{D}(z, 3)$ and

$$\operatorname{dens}(\mathcal{F}(f), \mathcal{D}(z, 3)) \geq \frac{1}{9} \operatorname{dens}(\mathcal{F}(f), \mathcal{D}(\zeta_k, 1)) \geq \frac{\eta_5}{9}.$$

Thus $\mathcal{J}(f)$ is thin at infinity. Hence Theorem A yields that $\mathcal{J}(f)$ has Lebesgue measure zero. $\square$

4.11 Examples

In this section we discuss some examples of functions satisfying the assumptions of Theorem B or Corollary C. Moreover, we give a result which says that certain slightly more restrictive conditions than those of Theorem B or Corollary C are stable under small perturbations of g.

Examples 4.11.1 and 4.11.2 are given in [Ber93b, §8].

Example 4.11.1. Let f be the Newton map of

$$g(z) = \int_0^z e^{at^d} dt,$$

with $a \in \mathbb{C} \setminus \{0\}$ and $d \geq 2$. Then the assumptions of Corollary C are satisfied so that $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$.

Example 4.11.2. Let f be the Newton map of

$$g(z) = \int_0^z e^{-t^2} dt + c,$$

with $c \in (-\sqrt{\pi}/2, \sqrt{\pi}/2)$. Then the assumptions of Corollary C are satisfied so that $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$.

Example 4.11.3. Let f be the Newton map of $g(z) = e^{z^d} - 1$, where $d \geq 2$. Then the assumptions of Corollary C are satisfied so that $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$.

To see this, note that

$$g'(z) = dz^{d-1} e^{z^d}, \quad g''(z) = dz^{d-2}(d - 1 + dz^d) e^{z^d}$$

and

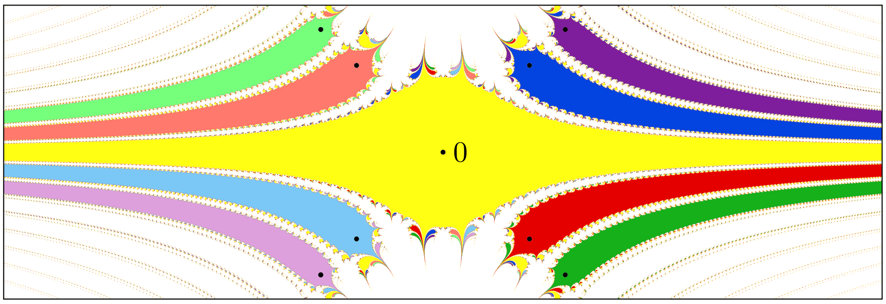
$$f(z) = z - \frac{e^{z^d} - 1}{dz^{d-1} e^{z^d}} = z \left(1 - \frac{e^{z^d} - 1}{dz^d e^{z^d}} \right). \tag{4.11.1}$$

Hence the zeros of g'' which are not zeros of g or g' are $z_j = \sqrt[d]{(d-1)/d} \cdot e^{(2j+1)\pi i/d}$ for $j \in \{1, ..., d\}$. Since the only zero of g', namely $z = 0$, is also a zero of g, the function f is in fact entire and has an attracting but not superattracting fixed point at zero. The attractive basin of zero contains a singular value of f and hence one of the points $z_1, ..., z_d$. Also, by (4.11.1), we have $f^n(e^{2\pi i/d}z) = e^{2\pi i/d} f^n(z)$ for all $z \in \mathbb{C}$, which implies that in fact $z_1, ..., z_d$ all lie in the attractive basin of zero. By Corollary C, $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$.

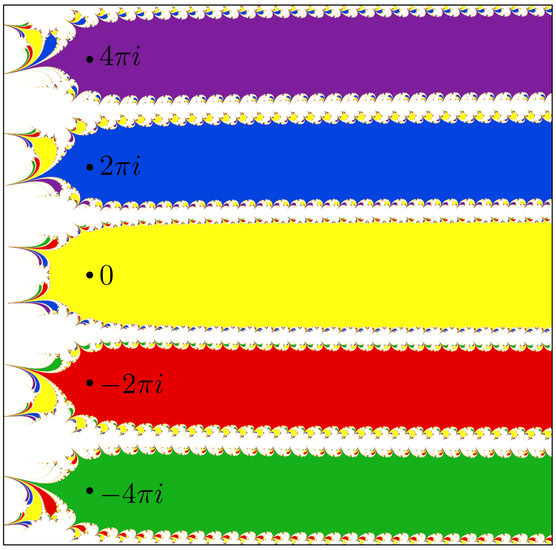
Moreover, the locally defined conjugate function $h(z) = f(\sqrt[d]{z})^d$ as considered in the proof of Theorem B satisfies

$$h(z) = z\left(1 - \frac{e^z - 1}{dze^z}\right)^d.$$

In particular, h does not depend on the chosen branch of $\sqrt[d]{\cdot}$ and extends to an entire function. Figure 4.7 shows attractive basins of f and h for $d = 2$, and Figure 4.8 displays attractive basins of f for $d = 3$.



(a) Attractive basins of f. The zeros of g are located at zero and $\sqrt{2k\pi}e^{(2j+1)\pi i/4}$ for $k \in \mathbb{N}$ and $j \in \{0, ..., 3\}$. The displayed range is $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 9, |\operatorname{Im} z| \leq 3\}$, and the picture shows only the attractive basins of zeros of g within this range.



(b) Attractive basins of h. The range shown is $\{z \in \mathbb{C} : -5 \leq \operatorname{Re} z \leq 27, |\operatorname{Im} z| \leq 5\pi\}$, and displayed are the attractive basins of fixed points of h which lie in this range.

Figure 4.7: The upper picture shows attractive basins of the function f from Newton's method for $g(z) = e^{z^2} - 1$ (cf. Example 4.11.3). The lower picture displays attractive basins of its conjugate $h(z) = f(\sqrt{z})^2$. Fixed points of f and h, respectively, are marked by black dots.

We now discuss another example in detail.

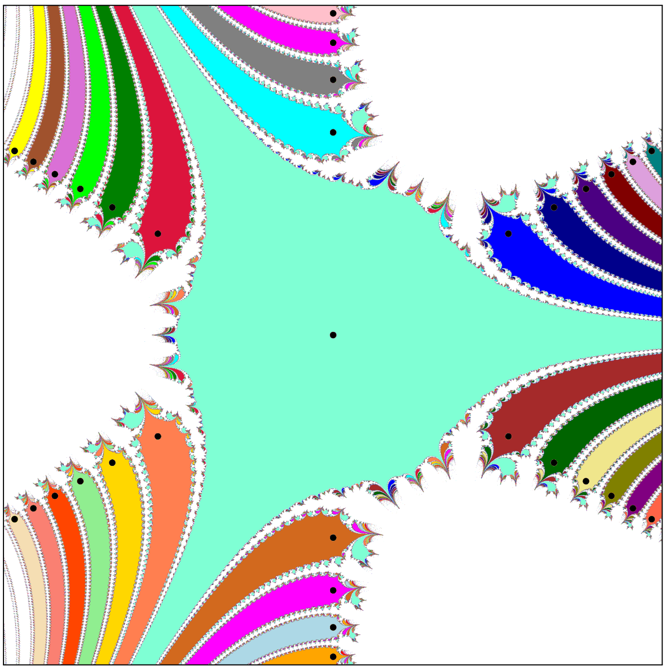


Figure 4.8: Attractive basins of the function f from Newton's method for $g(z) = e^{z^3} - 1$ (cf. Example 4.11.3). The range shown is $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 3, |\operatorname{Im} z| \leq 3\}$. Displayed are the attractive basins of zeros of g in this range. Zeros of g are marked by black dots.

Example 4.11.4. Let f be the function from Newton's method for

$$g(z) = (z^2 + 1)e^{-z^2} + c.$$

If $c \in [-1, 0)$ or $c = -7e^{-3/2}$, then $f^n(z)$ converges to zeros of g for almost all $z \in \mathbb{C}$. If $c \in [-15e^{-7/4}, -(17/2)e^{-5/4}]$, then the Julia set of f has Lebesgue measure zero, but there exists an open subset of $\mathbb{C}$ where $f^n(z)$ does not converge to any zero of g.

To see this, first note that

$$g'(z) = -2z^3 e^{-z^2}, \quad g''(z) = z^2(4z^2 - 6)e^{-z^2}.$$

So the zeros of g'' that are not zeros of g' are $-\sqrt{3/2}$ and $\sqrt{3/2}$. For $c \in \mathbb{R}$, we have $g(\mathbb{R}) \subset \mathbb{R}$, and g is increasing in $(-\infty, 0]$ and decreasing in $[0, \infty)$. Also, $\lim_{x \to \pm\infty} g(x) = c$ and $g(0) = 1 + c$.

If $c = -7e^{-3/2}$, then $f(\pm\sqrt{3/2}) = 0$ and $f^2(\pm\sqrt{3/2}) = \infty$. By Theorem B, $\mathcal{J}(f)$ has Lebesgue measure zero. Since $\pm\sqrt{3/2} \in \mathcal{J}(f)$, the same reasoning as in the proof of Bergweiler's result stated in Theorem 4.1.1 yields that each Fatou component of f must be the attractive basin of an attracting fixed point of f (cf. the remark in [Ber93b, p. 239]).

If $c = -1$, then $g(0) = g'(0) = 0$ so that f has an attracting but not superattracting fixed point at zero. The attractive basin of zero contains a singularity of f^{-1} and hence a zero of g'' that is not a zero of g'. Since $f(-z) = -f(z)$ for all $z \in \mathbb{C}$, both $-\sqrt{3/2}$ and $\sqrt{3/2}$ lie in the attractive basin of zero.

If $c \in (-1, 0)$, then the above considerations imply that g has two real zeros, given by x_0 and $-x_0$ for some $x_0 > 0$.

Suppose that $c \in (-1, -(5/2)e^{-3/2})$. Then $g(\sqrt{3/2}) < 0$. Because g is decreasing in $[0, \infty)$, this yields $x_0 \in (0, \sqrt{3/2})$. For $x > x_0$, we have $g(x) < 0$ and $g'(x) < 0$, and hence $f(x) < x$. Also, for $x \in (x_0, \sqrt{3/2})$, we have $g''(x) < 0$ and hence $f'(x) = g(x)g''(x)/g'(x)^2 > 0$. Thus f is increasing in $[x_0, \sqrt{3/2}]$. For $x \in (x_0, \sqrt{3/2}]$, we deduce that $x_0 < f(x) < x$ and hence $\lim_{n\to\infty} f^n(x) = x_0$. In particular, $\sqrt{3/2}$ lies in the attractive basin of x_0. Because $f(-z) = -f(z)$ for all $z \in \mathbb{C}$, also $-\sqrt{3/2}$ lies in the attractive basin of $-x_0$.

The case where $c \in (-(5/2)e^{-3/2}, 0)$ is analogous. For $c = -(5/2)e^{-3/2}$, we have $g(\pm\sqrt{3/2}) = 0$ so that $\pm\sqrt{3/2}$ are attracting fixed points of f.

Let us now show that for $c \in [-15e^{-7/4}, -(17/2)e^{-5/4}]$, there exists $x_0 > 0$ with $f(x_0) = -x_0$ and $|f'(x_0)| \leq 1$. Then $f^2(x_0) = f(-x_0) = -f(x_0) = x_0$ and $|(f^2)'(x_0)| = |f'(x_0)f'(-x_0)| = |f'(x_0)|^2 \leq 1$; that is, f has an attracting or indifferent periodic point of period two. In the indifferent case, the periodic point must be rationally indifferent, because $f'(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}$. The corresponding attractive or parabolic basin contains a singularity of f^{-1}, so it contains $\sqrt{3/2}$ or $-\sqrt{3/2}$. By symmetry, both $-\sqrt{3/2}$ and $\sqrt{3/2}$ are attracted by the cycle $\{-x_0, x_0\}$.

Let us now show that there is $x_0 > 0$ with $f(x_0) = -x_0$. This is equivalent to

$$(-4x_0^4 - x_0^2 - 1)e^{-x_0^2} = c. \tag{4.11.2}$$

The function $h(x) = (-4x^4 - x^2 - 1)e^{-x^2}$ satisfies $\lim_{x\to\pm\infty} h(x) = 0$, and it has local minima in $-\sqrt{7}/2$ and $\sqrt{7}/2$ with $h(\pm\sqrt{7}/2) = -15e^{-7/4}$ as well as a local maximum in 0 with $h(0) = -1$. Also, h is increasing in $[-\sqrt{7}/2, 0]$ and decreasing in $[0, \sqrt{7}/2]$. Thus, for $c \in [-15e^{-7/4}, -1]$, there exists $x_0 \in [0, \sqrt{7}/2]$ satisfying (4.11.2) and hence $f(x_0) = -x_0$. If $x_0 \in [\sqrt{5}/2, \sqrt{7}/2]$ and hence $|x_0^2 - 3/2| \leq 1/4$, then

$$|f'(x_0)| = \left|\frac{g(x_0)g''(x_0)}{g'(x_0)^2}\right| = \left|(x_0 - f(x_0))\frac{g''(x_0)}{g'(x_0)}\right| = 4|x_0^2 - 3/2| \leq 1.$$

Also, $h([\sqrt{5}/2, \sqrt{7}/2]) = [-15e^{-7/4}, -(17/2)e^{-5/4}]$, whence the claim follows. $\square$

Kriete [Kri01] studied convergence of Julia sets and stability of the behaviour of singular values for Newton's method for $g(z) = p(z)e^{q(z)} + az + b$ with polynomials p and q and $a, b \in \mathbb{C}$. Using his method, we can show that if g satisfies certain more restrictive assumptions than those of Theorem B or Corollary C, then these assumptions are also satisfied for functions that are in a suitable sense close to g. More precisely, we have the following result.

Lemma 4.11.5. *Let*

$$g(z) := \int_0^z p(t)e^{q(t)}dt + c,$$

where $p \not\equiv 0$ is a polynomial, q is a non-constant polynomial and $c \in \mathbb{C}$, and let f be the function from Newton's method for g. Suppose that g has only simple zeros, and that all zeros of g'' lie in attractive basins of attracting periodic cycles of f.

Let $(p_k), (q_k)$ be sequences of polynomials that converge locally uniformly in $\mathbb{C}$ to p and q, respectively, and satisfy $\deg(p_k) = \deg(p)$ and $\deg(q_k) = \deg(q)$ for all $k \in \mathbb{N}$. Also, let (c_k) be a sequence in $\mathbb{C}$ with $\lim_{k\to\infty} c_k = c$. For $k \in \mathbb{N}$, set

$$g_k(z) := \int_0^z p_k(t)e^{q_k(t)}dt + c_k,$$

and let f_k be the corresponding Newton map. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, all zeros of g_k'' lie in attractive basins of attracting periodic cycles of f_k. In particular, $\mathcal{J}(f_k)$ has Lebesgue measure zero for $k \geq k_0$.

If all zeros of g'' lie in attractive basins of attracting fixed points of f, then k_0 can be chosen such that for $k \geq k_0$, all zeros of g_k'' are in attractive basins of attracting fixed points of f_k. In particular, for $k \geq k_0$, the iterates $f_k^n(z)$ converges to zeros of g_k for almost all $z \in \mathbb{C}$ as $n \to \infty$.

The proof is based on the following lemma, which is stated in [Kri01, Lemma 11].

Lemma 4.11.6. *Let h be a transcendental meromorphic function, and let $\mathcal{A}$ denote the union of the attractive basins of its attracting periodic cycles. Let (h_k) be a sequence of transcendental meromorphic functions converging to h locally uniformly in $\mathbb{C}$. For all $k \in \mathbb{N}$, let $\mathcal{A}_k$ denote the union of the attractive basins of all attracting periodic cycles of h_k. Then for each compact subset $K \subset \mathcal{A}$, there exists $k_1 \in \mathbb{N}$ such that $K \subset \mathcal{A}_k$ for all $k \geq k_1$.*

Proof of Lemma 4.11.5. By the assumptions, (g_k), (g_k') and (g_k'') converge locally uniformly to g, g', and g'', respectively. Because g and g' do not have common zeros, also (f_k) converges to f locally uniformly. Denote by $z_1, ..., z_M$ the zeros of g''. Let $\varepsilon > 0$ such that the disks $D_j := \overline{\mathcal{D}(z_j, \varepsilon)}$ for $j \in \{1, ..., M\}$ are pairwise disjoint, and D_j is contained in an attractive basin of f for all $j \in \{1, ..., M\}$. By Hurwitz' theorem, for all large k, the functions g'' and g_k'' have the same number of zeros in each D_j counted with multiplicity. Since $\deg(p_k) = \deg(p)$ and $\deg(q_k) = \deg(q)$, the functions g'' and g_k'' have the same number of zeros in $\mathbb{C}$. Thus, for all large k, all zeros of g_k'' lie in $D := \bigcup_{j=1}^M D_j$. Also, by Lemma 4.11.6, D is contained in the union of attractive basins of f_k for all large k. This is the first claim.

Suppose now that all zeros of g'' lie in attractive basins of attracting fixed points of f. Denote these fixed points by $w_1, ..., w_N$. Let $\delta > 0$ such that the disks $D_j' := \overline{\mathcal{D}(w_j, \delta)}$ are pairwise disjoint, and D_j' is contained in the attractive basins of w_j for all $j \in \{1, ..., N\}$. Take k so large that for all $j \in \{1, ..., N\}$, the disk D_j' is contained in an attractive basin $A_{k,j}$ of f_k and $\sup_{z \in D_j'} |f(z) - f_k(z)| < \delta$. Then $f_k(w_j) \in D_j' \subset A_{k,j}$ and hence $A_{k,j}$ must be invariant. Moreover, there exists $n \in \mathbb{N}$ such that $f^n(D_j) \subset \bigcup_{l=1}^N \mathcal{D}(w_l, \delta/2)$. If k is sufficiently large, then $f_k^n(D_j) \subset \bigcup_{l=1}^N D_l' \subset \bigcup_{l=1}^N A_{k,l}$. So $f_k^n(z_j)$ converges to a fixed point of f_k as $n \to \infty$. $\square$

Chapter 5

Fatou and non-escaping sets of finite measure

This chapter concerns exponential polynomials f with the property that the Lebesgue measure of $\mathbb{C} \setminus (\mathcal{J}(f) \cap \mathcal{A}(f))$ is finite. More precisely, we prove a generalisation of Theorem D. This generalisation is stated in Theorem E in Section 5.1. In Section 5.2 we show that f can be approximated by simpler functions in large parts of the complex plane, and use this to prove that $|f(z)|$ is large outside a set of finite Lebesgue measure. Then, in Section 5.3, we show that f is injective in certain small disks. We complete the proof of Theorem E in Section 5.4, using a construction of McMullen [McM87] that has been applied by various authors. We also use ideas from [Six15]. Finally, Section 5.5 concerns examples illustrating the necessity of the assumptions of Theorem E.

5.1 Main result

The main objective of this chapter is to prove the following generalisation of Theorem D.

Theorem E. *Let*

$$f(z) := \sum_{j=1}^{N} Q_j(z) \exp(b_j z^d + P_j(z)),$$

where $d \in \mathbb{N}$ *with* $d \geq 3$*,* P_j *and* Q_j *are polynomials with* $Q_j \not\equiv 0$ *and* $\deg(P_j) < d$*, and* $b_j \in \mathbb{C} \setminus \{0\}$ *are distinct numbers satisfying* $\arg(b_j) \leq \arg(b_{j+1}) \leq \arg(b_j) + \pi$ *for all* $j \in \{1, ..., N-1\}$ *and* $\arg(b_N) \geq \arg(b_1) + \pi$*, with the argument chosen in* $[0, 2\pi)$*.*

If there exists $j \in \{1, ..., N-1\}$ *such that* $\arg(b_{j+1}) = \arg(b_j) + \pi$*, or if* $\arg(b_N) = \arg(b_1) + \pi$*, in addition suppose that there are* $k, l \in \{1, ..., N\}$ *with* $\arg(b_k) = \arg(b_j)$ *and* $\arg(b_l) = \arg(b_{j+1})$*, or* $\arg(b_k) = \arg(b_1)$ *and* $\arg(b_l) = \arg(b_N)$*, respectively, such that*

$$\deg\left(P_k - \frac{b_k}{b_l} P_l\right) \leq d - 3. \tag{5.1.1}$$

Then the Lebesgue measure of $\mathbb{C} \setminus (\mathcal{A}(f) \cap \mathcal{J}(f))$ *is finite.*

Note that the conditions on the b_j imply that $N \geq 2$. The integers k, l are introduced because several of the numbers b_j may have the same argument. The additional assumption (5.1.1) in the case when $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, ..., N-1\}$ or $\arg(b_N) = \arg(b_1) + \pi$ cannot be left out, as the following example shows.

Example 5.1.1. Let

$$h(z) := \frac{1}{2}\exp(z^3 + iz) - \frac{1}{2}\exp(-z^3 + iz) = \exp(iz)\sinh(z^3).$$

Then h has a superattracting fixed point at zero, and the attractive basin of zero has infinite Lebesgue measure. In particular, the Lebesgue measure of $\mathbb{C} \setminus (\mathcal{A}(h) \cap \mathcal{J}(h))$ is infinite.

5.2 The behaviour of f

Throughout Sections 5.2-5.4, that is, until the end of the proof of Theorem E, let f denote an entire function satisfying the assumptions of Theorem E.

In this section we prove several properties of the function f. We first introduce some notation. For j, $k \in \{1, ..., N\}$ with $j \neq k$, let

$$P_{j,k}(z) := (b_j z^d + P_j(z)) - (b_k z^d + P_k(z)) = (b_j - b_k)z^d + (P_j(z) - P_k(z)).$$

Set

$$\nu := d - \frac{5}{2},$$

and define sets

$$\mathcal{U}_1 := \{w \in \mathbb{C} : |\operatorname{Re} w| < |w|^{\nu/d}\}$$

and

$$\mathcal{U}_2 := \{w \in \mathbb{C} : |\operatorname{Re} w| < 2|w|^{\nu/d}\}.$$

Moreover, define 'exceptional sets'

$$\mathcal{E}_l := \bigcup_{\substack{j,k=1 \\ j \neq k}}^{N} P_{j,k}^{-1}(\mathcal{U}_l),$$

for $l \in \{1, 2\}$ (see Figure 5.1). We will prove later that the function f behaves 'nicely' outside $\mathcal{E}_1$.

Lemma 5.2.1. *Let P be a polynomial of degree d, with d as before. Then the Lebesgue measure of $P^{-1}(\mathcal{U}_2)$ is finite.*

Proof. Write

$$P(z) = \sum_{j=0}^{d} a_j z^j.$$

Fix $R > 2^{2d/5}$ such that all critical values of P are contained in $\mathcal{D}(0, R)$, and let V be a component of $P^{-1}(\mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup (-\infty, 0]))$. By Lemma 2.2.2, P maps V conformally onto $\mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup (-\infty, 0])$. Let $\varphi : \mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup (-\infty, 0]) \to V$ denote the corresponding inverse function. Then $P^{-1}(\mathcal{U}_2) \cap V = \varphi(\mathcal{U}_2 \setminus \overline{\mathcal{D}(0, R)})$. Set $\eta(r) := \arcsin(2r^{-5/(2d)})$ and

$$W := \mathcal{U}_2 \setminus \overline{\mathcal{D}(0, R)} = \left\{ re^{i\theta} : r > R, \min\left\{ \left|\theta - \frac{\pi}{2}\right|, \left|\theta - \frac{3\pi}{2}\right| \right\} < \eta(r) \right\}.$$

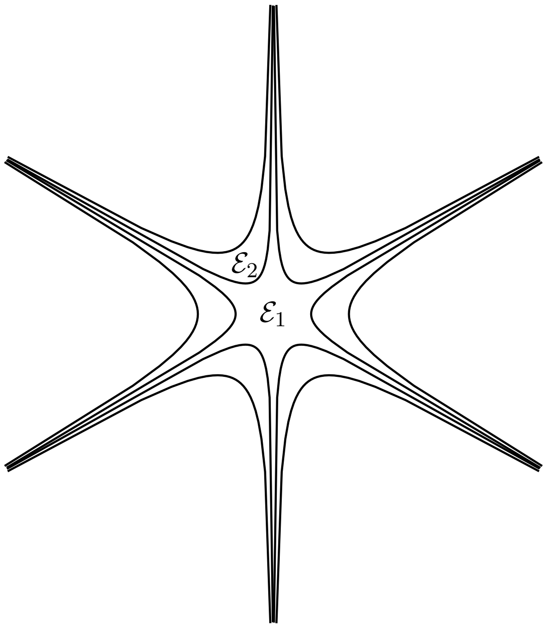


Figure 5.1: The sets $\mathcal{E}_1$ (bounded by the inner curves) and $\mathcal{E}_2$ (bounded by the outer curves) for $f(z) = Q_1(z)e^{(1/2)z^3} + Q_2(z)e^{-(1/2)z^3}$.

We have

$$\begin{aligned}\operatorname{meas}(P^{-1}(\mathcal{U}_2) \cap V) &= \int_W |\varphi'(z)|^2 dx\, dy = \int_W \frac{1}{|P'(\varphi(z))|^2} dx\, dy \\ &= \int_R^\infty r \left(\int_{\pi/2-\eta(r)}^{\pi/2+\eta(r)} \frac{1}{|P'(\varphi(re^{i\theta}))|^2} d\theta + \int_{3\pi/2-\eta(r)}^{3\pi/2+\eta(r)} \frac{1}{|P'(\varphi(re^{i\theta}))|^2} d\theta \right) dr. \quad (5.2.1)\end{aligned}$$

If r is sufficiently large, then

$$r = |P(\varphi(re^{i\theta}))| \leq 2^{d/(d-1)} |a_d| |\varphi(re^{i\theta})|^d.$$

Thus

$$|\varphi(re^{i\theta})| \geq 2^{-1/(d-1)} |a_d|^{-1/d} \cdot r^{1/d}$$

and

$$|P'(\varphi(re^{i\theta}))| \geq \frac{1}{2} d |a_d| |\varphi(re^{i\theta})|^{d-1} \geq \frac{1}{4} d |a_d|^{1/d} \cdot r^{(d-1)/d}. \quad (5.2.2)$$

Since $\arcsin(x) \leq (\pi/2)x$ for $x \in [0,1]$, we also have

$$\eta(r) = \arcsin(2r^{-5/(2d)}) \leq \pi r^{-5/(2d)}. \quad (5.2.3)$$

Inserting (5.2.2) and (5.2.3) into (5.2.1) yields

$$\begin{aligned}\operatorname{meas}(P^{-1}(\mathcal{U}_2) \cap V) &\leq \int_R^\infty r \cdot 4\eta(r) \cdot \frac{16}{d^2 |a_d|^{2/d} r^{2(d-1)/d}} dr \\ &\leq \frac{64\pi}{d^2} |a_d|^{-2/d} \int_R^\infty r \cdot r^{-5/(2d)} \cdot r^{-2(d-1)/d} dr \\ &= \frac{64\pi}{d^2} |a_d|^{-2/d} \int_R^\infty r^{-(1+1/(2d))} dr < \infty.\end{aligned}$$

Since there are only finitely many such components V and $P^{-1}(\overline{\mathcal{D}(0,R)})$ has finite Lebesgue measure, the claim follows. □

The following corollary is an immediate consequence of Lemma 5.2.1.

Corollary 5.2.2. *The Lebesgue measure of* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *is finite.*

The next lemma yields that if $R_0 > 0$ is large, then in each component of $\mathbb{C} \setminus (\mathcal{E}_1 \cup \mathcal{D}(0, R_0))$, the function f behaves like one of its summands $Q_m(z)\exp(b_m z^d + P_m(z))$.

Lemma 5.2.3. *Let* $\varepsilon > 0$. *Then there exists* $R_0 > 0$ *such that for each connected component* V *of* $\mathbb{C} \setminus (\mathcal{E}_1 \cup \mathcal{D}(0, R_0))$, *there is* $m \in \{1, ..., N\}$ *such that for all* $z \in V$ *and all* $j \in \{1, ..., N\}$ *with* $j \neq m$, *we have*

$$\operatorname{Re}(b_m z^d + P_m(z)) > \operatorname{Re}(b_j z^d + P_j(z))$$

and

$$\left| \frac{f(z)}{Q_m(z)\exp(b_m z^d + P_m(z))} - 1 \right| < \varepsilon,$$
$$\left| \frac{f'(z)}{d b_m z^{d-1} Q_m(z)\exp(b_m z^d + P_m(z))} - 1 \right| < \varepsilon,$$
$$\left| \frac{f''(z)}{d^2 b_m^2 z^{2d-2} Q_m(z)\exp(b_m z^d + P_m(z))} - 1 \right| < \varepsilon.$$

Proof. Fix $R_0 > 0$ such that all zeros of the polynomials $P_{j,k}$ are contained in $\mathcal{D}(0, R_0)$, and let V be a component of $\mathbb{C} \setminus (\mathcal{E}_1 \cup \mathcal{D}(0, R_0))$. Take $z_0 \in V$, and let $m \in \{1, ..., N\}$ such that

$$\operatorname{Re}(b_m z_0^d + P_m(z_0)) = \max_{1 \leq j \leq N} \operatorname{Re}(b_j z_0^d + P_j(z_0)). \tag{5.2.4}$$

Let $j \in \{1, ..., N\}$ with $j \neq m$. For all $z \in V$, we have $P_{m,j}(z) \notin \mathcal{U}_1$ and hence

$$|\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z))| = |\operatorname{Re} P_{m,j}(z)| \geq |P_{m,j}(z)|^{\nu/d}. \tag{5.2.5}$$

In particular,

$$\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z)) \neq 0$$

for all $z \in V$. By continuity and (5.2.4), we deduce that

$$\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z)) > 0$$

for all $z \in V$, and hence

$$\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z)) \geq |P_{m,j}(z)|^{\nu/d}$$

by (5.2.5). Thus, for all $z \in V$,

$$\begin{aligned}
&\left| \frac{f(z)}{Q_m(z)\exp(b_m z^d + P_m(z))} - 1 \right| \\
&= \left| \frac{\sum_{j=1}^N Q_j(z)\exp(b_j z^d + P_j(z)) - Q_m(z)\exp(b_m z^d + P_m(z))}{Q_m(z)\exp(b_m z^d + P_m(z))} \right| \\
&= \left| \sum_{\substack{j=1 \\ j \neq m}}^N \frac{Q_j(z)}{Q_m(z)} \exp(b_j z^d + P_j(z) - (b_m z^d + P_m(z))) \right| \\
&\leq \sum_{\substack{j=1 \\ j \neq m}}^N \left| \frac{Q_j(z)}{Q_m(z)} \right| \exp(\operatorname{Re}(b_j z^d + P_j(z)) - \operatorname{Re}(b_m z^d + P_m(z))) \\
&\leq \sum_{\substack{j=1 \\ j \neq m}}^N \left| \frac{Q_j(z)}{Q_m(z)} \right| \exp(-|P_{m,j}(z)|^{\nu/d}) < \varepsilon
\end{aligned}$$

if R_0 and hence $|z|$ is sufficiently large. This is the result for f.

Moreover,

$$f'(z) = \sum_{j=1}^{N} (db_j z^{d-1} Q_j(z) + P_j'(z) Q_j(z) + Q_j'(z)) \exp(b_j z^d + P_j(z)).$$

The result for f' now follows from similar estimates as above and the fact that

$$db_j z^{d-1} Q_j(z) + P_j'(z) Q_j(z) + Q_j'(z) = (1 + o(1)) db_j z^{d-1} Q_j(z)$$

as $|z| \to \infty$. Analogously, the result for f'' follows from the fact that

$$f''(z) = \sum_{j=1}^{N} (1 + o(1)) d^2 b_j^2 z^{2d-2} Q_j(z) \exp(b_j z^d + P_j(z))$$

as $|z| \to \infty$. □

Remark 5.2.4. In order to prove Lemma 5.2.3, we did not need any assumptions on the arguments of the numbers b_j. In particular, the conclusion remains true without the additional condition (5.1.1) in the case when $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, ..., N-1\}$ or $\arg(b_N) = \arg(b_1) + \pi$. This is different for the next result.

Lemma 5.2.5. *Let $\alpha \in (0, \nu)$. Then there exists $R_1 > 0$ such that for all $z \in \mathbb{C} \setminus \mathcal{E}_1$ with $|z| \geq R_1$, we have*

$$|f(z)| \geq \exp(|z|^\alpha) \quad \textit{and} \quad |f'(z)| \geq \exp(|z|^\alpha).$$

Remark 5.2.6. The conclusion of Lemma 5.2.5 is not true in general without the additional condition (5.1.1) in the case when $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, ..., N\}$ or $\arg(b_N) = \arg(b_1) + \pi$. We will prove in Section 5.5 that the function $h(z) = (1/2)\exp(z^3 + iz) - (1/2)\exp(-z^3 + iz)$ given in Example 5.1.1 is bounded in a set of infinite Lebesgue measure.

Proof of Lemma 5.2.5. We prove the statement for f. The proof for f' is analogous. Let $z \in \mathbb{C} \setminus \mathcal{E}_1$. If $|z|$ is sufficiently large, Lemma 5.2.3 yields $m \in \{1, ..., N\}$ such that

$$\operatorname{Re}(b_m z^d + P_m(z)) > \operatorname{Re}(b_j z^d + P_j(z))$$

for all $j \in \{1, ..., N\}$ with $j \neq m$ and

$$\left| \frac{f(z)}{Q_m(z) \exp(b_m z^d + P_m(z))} - 1 \right| \leq \frac{1}{2}.$$

Then

$$|f(z)| \geq \frac{1}{2} |Q_m(z)| \exp(\operatorname{Re}(b_m z^d + P_m(z))).$$

Thus it suffices to show that there exists $j \in \{1, ..., N\}$ with

$$\operatorname{Re}(b_j z^d + P_j(z)) \geq 2|z|^\alpha.$$

We first consider the case where f satisfies the assumptions of Theorem D, that is,

$$\arg(b_{j+1}) < \arg(b_j) + \pi \text{ for all } j \in \{1, ..., N-1\} \text{ and } \arg(b_N) > \arg(b_1) + \pi.$$

Then there is a constant $C > 0$ such that for all $w \in \mathbb{C}$, there exists $j \in \{1, ..., N\}$ with $\operatorname{Re}(b_j w^d) \geq 2C|w|^d$. In particular, this applies to $w = z$. Since $\deg(P_j) < d$, we deduce that

$$\operatorname{Re}(b_j z^d + P_j(z)) \geq 2C|z|^d - |P_j(z)| \geq C|z|^d > 2|z|^\alpha$$

if $|z|$ is sufficiently large.

Now suppose that f does not satisfy the assumptions of Theorem D. Then the assumptions of Theorem E imply that there are $k, l \in \{1, ..., N\}$ such that $|\arg(b_k) - \arg(b_l)| = \pi$ and the polynomial $h := P_k - (b_k/b_l)P_l$ satisfies $\deg(h) \leq d - 3$. For $g := (1/b_l)P_l$, we have

$$P_l = b_l g \quad \text{and} \quad P_k = b_k g + h.$$

Moreover, with $\beta_k := \arg(b_k)$, we have

$$b_k = |b_k| e^{i\beta_k} \quad \text{and} \quad b_l = -|b_l| e^{i\beta_k}.$$

Thus

$$b_k - b_l = (|b_k| + |b_l|) e^{i\beta_k}$$

and

$$P_{k,l}(z) = (|b_k| + |b_l|) e^{i\beta_k} (z^d + g(z)) + h(z).$$

Assume without loss of generality that

$$\operatorname{Re}(b_k(z^d + g(z))) \geq \operatorname{Re}(b_l(z^d + g(z))).$$

Then $\operatorname{Re}(b_k(z^d + g(z))) \geq 0$. Since $z \notin \mathcal{E}_1$, we get

$$|P_{k,l}(z)|^{\nu/d} \leq |\operatorname{Re} P_{k,l}(z)| \leq (|b_k| + |b_l|) \operatorname{Re}(e^{i\beta_k}(z^d + g(z))) + |h(z)|.$$

Hence

$$\begin{aligned}
\operatorname{Re}(b_k z^d + P_k(z)) &= \operatorname{Re}(b_k(z^d + g(z)) + h(z)) \\
&= |b_k| \operatorname{Re}(e^{i\beta_k}(z^d + g(z))) + \operatorname{Re}(h(z)) \\
&\geq \frac{|b_k|}{|b_k| + |b_l|} (|P_{k,l}(z)|^{\nu/d} - |h(z)|) - |h(z)|.
\end{aligned}$$

Because

$$|P_{k,l}(z)| \geq 2^{-d/\nu} (|b_k| + |b_l|) |z|^d$$

if $|z|$ is large and since $\deg(h) < \nu$, we deduce that

$$\operatorname{Re}(b_k z^d + P_k(z)) \geq \frac{1}{4} |b_k| (|b_k| + |b_l|)^{(\nu/d)-1} |z|^\nu > 2|z|^\alpha$$

if $|z|$ is sufficiently large. □

5.3 Injectivity

In this section we prove that f is injective in certain disks contained in $\mathbb{C} \setminus \mathcal{E}_1$.

Lemma 5.3.1. *Let $\sigma \in (0, 1/(4d \max_j |b_j|))$. There exists $R_2 > 0$ such that the following holds.*

If $z \in \mathbb{C} \setminus \mathcal{E}_1$ with $|z| \geq R_2$ is such that $\mathcal{D}(z, 2\sigma|z|^{-(d-1)}) \subset \mathbb{C} \setminus \mathcal{E}_1$, then f is injective in $\mathcal{D}(z, 2\sigma|z|^{-(d-1)})$.

Proof. Let $r := 2\sigma|z|^{-(d-1)}$. By Lemma 5.2.3, there exists $m \in \{1, ..., N\}$ such that

$$\sup_{|\zeta - z|<r} \left|\frac{f''(\zeta)}{f'(\zeta)}\right| \le \frac{3}{2} d|b_m| \sup_{|\zeta - z|<r} |\zeta|^{d-1} \le 2d|b_m||z|^{d-1} < \frac{1}{r}$$

if $|z|$ is sufficiently large. Thus f is injective in $\mathcal{D}(z, r)$ by Lemma 2.2.4. □

We also require the following lemma.

Lemma 5.3.2. *There are $C_1, R_3 > 0$ such that for all $z \in \mathbb{C} \setminus \mathcal{E}_2$ with $|z| \ge R_3$, we have*

$$\operatorname{dist}(z, \mathcal{E}_1) \ge C_1|z|^{-3/2}.$$

The following corollary is an immediate consequence of Lemmas 5.3.2 and 5.3.1.

Corollary 5.3.3. *Let σ be as in Lemma 5.3.1. There is $R_4 > 0$ such that if $z \in \mathbb{C} \setminus \mathcal{E}_2$ and $|z| \ge R_4$, then f is injective in $\mathcal{D}(z, 2\sigma|z|^{-(d-1)})$.*

Proof of Lemma 5.3.2. Let $z \in \mathbb{C} \setminus \mathcal{E}_2$. It suffices to show that if $|z|$ is sufficiently large, then

$$\operatorname{dist}(z, \mathcal{E}_1 \cap \mathcal{D}(z, 1)) \ge C_1|z|^{-3/2}$$

for some constant $C_1 > 0$. Let $w \in \mathcal{E}_1 \cap \mathcal{D}(z, 1)$. Then there are $j, k \in \{1, ..., N\}$ with $j \ne k$ such that $|\operatorname{Re}(P_{j,k}(w))| \le |P_{j,k}(w)|^{\nu/d}$. If $|z|$ is sufficiently large, then $|w| \le (6/5)^{1/\nu}|z|$ and

$$\begin{aligned} |P_{j,k}(z) - P_{j,k}(w)| &\ge |\operatorname{Re}(P_{j,k}(z))| - |\operatorname{Re}(P_{j,k}(w))| \\ &\ge 2|P_{j,k}(z)|^{\nu/d} - |P_{j,k}(w)|^{\nu/d} \\ &\ge 2 \cdot \frac{4}{5}|b_j - b_k|^{\nu/d}|z|^{\nu} - \frac{6}{5}|b_j - b_k|^{\nu/d}|w|^{\nu} \\ &\ge \frac{8}{5}|b_j - b_k|^{\nu/d}|z|^{\nu} - \left(\frac{6}{5}\right)^2 |b_j - b_k|^{\nu/d}|z|^{\nu} \\ &= \frac{4}{25}|b_j - b_k|^{\nu/d}|z|^{d-5/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |P_{j,k}(z) - P_{j,k}(w)| = \left|\int_w^z P'_{j,k}(\zeta)\, d\zeta\right| &\le \sup_{|\zeta - z|<1} |P'_{j,k}(\zeta)| \cdot |z - w| \\ &\le 2d|b_j - b_k|(|z| + 1)^{d-1}|z - w| \ \le 4d|b_j - b_k||z|^{d-1}|z - w| \end{aligned}$$

if $|z|$ is sufficiently large. Thus

$$|z - w| \ge \frac{|b_j - b_k|^{\nu/d-1}}{25d}|z|^{-3/2} \ge C_1|z|^{-3/2}$$

for $C_1 := \min_{l \ne n} |b_l - b_n|^{\nu/d-1}/(25d)$. □

5.4 Proof of Theorem E

In this section we prove Theorem E. We require the following lemma due to Zheng [Zhe06, Corollary 6 and Remark (J)].

Lemma 5.4.1. *Let g be a transcendental entire function of the form*

$$g(z) = \sum_{j=1}^{M} q_j(z) \exp(p_j(z)),$$

with polynomials p_j and q_j. Then the Fatou set of g has no multiply connected components.

We make some comments on the proof of Lemma 5.4.1. Each multiply connected Fatou component of a transcendental entire function is a wandering domain [Bak84, Theorem 3.1]. Lemma 5.4.1 is deduced from a result that rules out multiply connected wandering domains for functions satisfying a certain growth condition [Zhe06, Corollary 5]. More precisely, the result says that if g is an entire function, and there exists $a > 1$ such that

$$\log M(2r, g) > a \log M(r, g)$$

or

$$T(2r, g) > aT(r, g)$$

for all large r, then g has no multiply connected wandering domains. Here, $T(r, g)$ denotes the Nevanlinna characteristic of g. We do not go into details about Nevanlinna theory here, and only mention that since g is entire, we have

$$T(r, g) = 1/(2\pi) \int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta,$$

where $\log^+ x = \max\{x, 0\}$. See also [Ste78, Satz 1] for the computation of the Nevanlinna characteristic of exponential polynomials.

Proof of Theorem E. Throughout the proof, write

$$\mathcal{J}_A := \mathcal{J}(f) \cap \mathcal{A}(f).$$

The strategy of the proof is as follows. First, we construct a suitable collection $\mathcal{S}$ of squares in $\mathbb{C} \setminus \mathcal{E}_1$ such that $\mathbb{C} \setminus \bigcup_{S \in \mathcal{S}} S$ has finite Lebesgue measure. For $S_0 \in \mathcal{S}$, we then show that points in S_0 which stay in $\bigcup_{S \in \mathcal{S}} S$ under iteration lie in $\mathcal{J}_{\mathcal{A}}$, and that the set of such points has large density in S_0. Finally, we prove that $\sum_{S \in \mathcal{S}} \text{meas}(S \setminus \mathcal{J}_{\mathcal{A}}) < \infty$.

Let $\varepsilon = 1/3$, $\alpha \in (0, \nu)$, $\sigma \in (0, 1/(4d \max_j |b_j|))$ and $\beta > d$; and let R_0, R_1, R_2, x_0 be the corresponding constants from Lemmas 5.2.3, 5.2.5, 5.3.1 and 2.6.1, respectively. Also, let C_1 and R_3 be the constants from Lemma 5.3.2. Let B_0 be an open square centred at zero with sides parallel to the real and imaginary axis such that all $z \in \mathbb{C} \setminus B_0$ satisfy

$$|z| > \max\{R_0, ..., R_3, x_0\},$$

$$\frac{C_1}{|z|^{3/2}} > \frac{3\sigma}{|z|^{d-1}}, \tag{5.4.1}$$

$$M(|z|, f) \leq E_\beta(|z|) \tag{5.4.2}$$

and

$$\frac{1}{2} d \min_k |b_k| |z|^d > 2. \tag{5.4.3}$$

We will later impose further lower bounds for the side length of B_0. For a compact set $X \subset \mathbb{C}$, set

$$\mathcal{M}(X) := \max_{z \in X} |z| \quad \text{and } \mu(X) := \min_{z \in X} |z|.$$

More generally, for a continuous function $g : X \to \mathbb{C}$, let

$$\mathcal{M}(X, g) := \max_{z\in X} |g(z)| \quad \text{and } \mu(X, g) := \min_{z\in X} |g(z)|.$$

Let $\tilde{\mathcal{S}}$ be a collection of closed squares in $\mathbb{C}$ with sides parallel to the real and imaginary axis such that

- $\bigcup_{S\in\tilde{\mathcal{S}}} S = \mathbb{C} \setminus B_0$;
- all $S_1, S_2 \in \tilde{\mathcal{S}}$ with $S_1 \neq S_2$ have disjoint interior;
- for all $S \in \tilde{\mathcal{S}}$, the side length s of S satisfies

$$\frac{\sigma}{4\sqrt{2}\mu(S)^{d-1}} \leq s \leq \frac{\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}}.$$

If the side length of B_0 is sufficiently large, this can be achieved as follows. First, divide $\mathbb{C} \setminus B_0$ into squares of a fixed size so that the side length of all squares satisfies the lower bound. If the side length s of a square does not satisfy the upper bound, divide it into four squares of side length $s/2$, and then continue this procedure until the upper bound is satisfied.

Let $\mathcal{S}$ be the collection of all $S \in \tilde{\mathcal{S}}$ such that $\operatorname{dist}(S, \mathcal{E}_1) > 2\sigma/\mu(S)^{d-1}$. By Lemma 5.3.2, the definition of $\mathcal{S}$ and (5.4.1),

$$\mathbb{C} \setminus (\mathcal{E}_2 \cup B_0) \subset \bigcup_{S\in\mathcal{S}} S \subset \mathbb{C} \setminus (\mathcal{E}_1 \cup B_0). \tag{5.4.4}$$

Next, we construct a subset of $\mathcal{J}_{\mathcal{A}}$ as an intersection of nested sets. Fix a square $S_0 \in \mathcal{S}$. Set

$$\mathcal{K}_0 := \{S_0\}$$

and for $n \in \mathbb{N}$, let

$$\mathcal{K}_n := \{T_n \subset S_0 : \, f^n(T_n) \in \mathcal{S} \text{ and } T_n \subset T_{n-1} \text{ for some } T_{n-1} \in \mathcal{K}_{n-1}\}.$$

We first show that

$$T := \bigcap_{n\geq 0} \left(\bigcup_{T_n\in\mathcal{K}_n} T_n \right) \subset \mathcal{J}_{\mathcal{A}}.$$

To do so, let $z \in T$. Then $f^n(z) \in \mathbb{C} \setminus \mathcal{E}_1$ for all $n \geq 0$. By Lemmas 5.2.5 and 2.6.1 and (5.4.2), we have

$$|f^n(z)| \geq E_\alpha^n(|z|) \geq E_\beta^{n-2}(|z|) \geq M^{n-2}(|z|, f)$$

for all $n \geq 4$. This yields $z \in \mathcal{A}(f)$. For $n \in \mathbb{N}$, let $z_n := f^n(z)$. By Lemma 5.2.3 and (5.4.3),

$$\left| z_n \frac{f'(z_n)}{f(z_n)} \right| \geq \frac{1}{2} d \min_k |b_k| |z_n|^d > 2$$

if $|z|$ is sufficiently large. Using Lemmas 2.4.18 and 5.4.1, we deduce that $z \in \mathcal{J}(f)$. Thus $T \subset \mathcal{J}_{\mathcal{A}}$.

For $A \subset \mathbb{C}$, define

$$\operatorname{pack}(A) := \{S \in \mathcal{S} : \, S \subset A\}.$$

We will show that for any square $S \in \mathcal{S}$, we have

$$\operatorname{dens}\left(f(S) \setminus \bigcup_{S'\in\operatorname{pack}(f(S))} S', \, f(S) \right) \leq \exp\left(-\frac{1}{2}\mu(S)^\alpha \right).$$

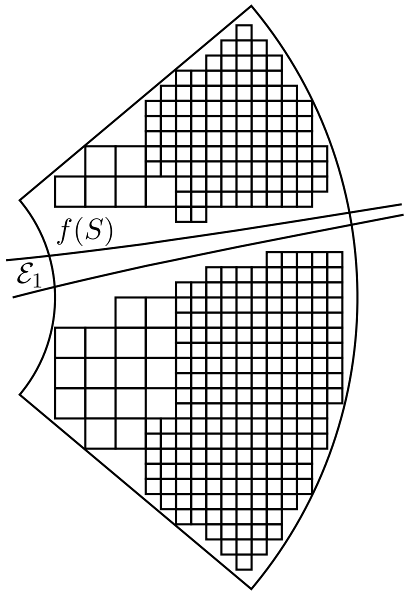


Figure 5.2: An illustration of $\mathrm{pack}(f(S))$ (not to scale).

See Figure 5.2 for an illustration of $\mathrm{pack}(f(S))$.

By Lemma 5.3.1, f is injective in S. We have

$$\mathrm{meas}(f(S)) = \int_S |f'(z)|^2 dx\, dy \geq \mathrm{meas}(S) \cdot \mu(S, f')^2.$$

Moreover, by Lemma 5.2.5, we have $|f(z)| \geq \exp(|z|^\alpha)$ for all $z \in S$, and hence

$$f(S) \subset \mathbb{C} \setminus B_0 = \bigcup_{S' \in \tilde{\mathcal{S}}} S'.$$

By (5.4.4) and Corollary 5.2.2, the Lebesgue measure of the union of all squares $S' \in \tilde{\mathcal{S}} \setminus \mathcal{S}$ is at most $\mathrm{meas}(\mathcal{E}_2) < \infty$. We now consider the union of all squares $S' \in \mathcal{S}$ with $S' \cap \partial f(S) \neq \emptyset$. The length of $\partial f(S)$ satisfies

$$\mathrm{length}(\partial f(S)) = \int_{\partial S} |f'(z)|\, |dz| \leq \mathcal{M}(S, f')\, \mathrm{length}(\partial S) \leq \frac{4\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}} \mathcal{M}(S, f').$$

Analogously,

$$\mathrm{length}(\partial f(S)) \geq \frac{\sigma}{\sqrt{2}\mu(S)^{d-1}} \mu(S, f').$$

By Lemma 5.2.5, $|f'(z)| \geq \exp(|z|^\alpha)$ for all $z \in S$. In particular, $\mathrm{length}(\partial f(S)) > 1$. For $S' \in \mathcal{S}$ with $S' \cap \partial f(S) \neq \emptyset$, we have

$$S' \subset \{z \in \mathbb{C} : \mathrm{dist}(z, \partial f(S)) \leq 1\}.$$

By Lemma 2.3.1,

$$\mathrm{meas}(\{z \in \mathbb{C} : \mathrm{dist}(z, \partial f(S)) \leq 1\}) \leq \frac{9\pi}{2} \cdot \mathrm{length}(\partial f(S)) \leq \frac{18\pi\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}} \mathcal{M}(S, f').$$

Altogether, we get

$$\mathrm{meas}\left(f(S) \setminus \bigcup_{S' \in \mathrm{pack}(f(S))} S' \right) \leq \mathrm{meas}(\mathcal{E}_2) + \frac{18\pi\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}} \mathcal{M}(S, f') \leq \mathcal{M}(S, f')$$

if the side length of B_0 and hence $\mathcal{M}(S)$ is sufficiently large. Thus

$$\operatorname{dens}\left(f(S) \setminus \bigcup_{S' \in \operatorname{pack}(f(S))} S',\, f(S)\right) \le \frac{\mathcal{M}(S, f')}{\operatorname{meas}(S) \cdot \mu(S, f')^2}.$$

Let z_0 be the centre of S. Then $S \subset \mathcal{D}(z_0, (\sigma/2)|z_0|^{-(d-1)})$, and by Lemma 5.3.1, f is injective in $\mathcal{D}(z_0, 2\sigma|z_0|^{-(d-1)})$. By the Koebe distortion theorem and Lemma 5.2.5,

$$\begin{aligned}\operatorname{dens}\left(f(S) \setminus \bigcup_{S' \in \operatorname{pack}(f(S))} S',\, f(S)\right) &\le \left(\frac{5}{3}\right)^4 \frac{1}{\operatorname{meas}(S)\mu(S, f')} \\ &\le \left(\frac{5}{3}\right)^4 \frac{32\mu(S)^{2(d-1)}}{\sigma^2 \exp(\mu(S)^\alpha)} \\ &\le \exp\left(-\frac{1}{2}\mu(S)^\alpha\right) \qquad (5.4.5)\end{aligned}$$

if $\mu(S)$ is sufficiently large.

Recall that we want to show that the density of $\mathcal{J}_\mathcal{A}$ in the square $S_0 \in \mathcal{S}$ is large by considering the subset $T = \bigcap_{n \ge 0} \bigcup_{T_n \in \mathcal{K}_n} T_n \subset S_0 \cap \mathcal{J}_\mathcal{A}$. Let $n \ge 0$ and $T_n \in \mathcal{K}_n$. Then $f^n(T_n) \in \mathcal{S}$. By (5.4.5) applied to $S = f^n(T_n)$ and Lemma 5.2.5,

$$\begin{aligned}&\operatorname{dens}\left(f^{n+1}(T_n) \setminus \bigcup_{S' \in \operatorname{pack}(f^{n+1}(T_n))} S',\, f^{n+1}(T_n)\right) \\ &\le \exp\left(-\frac{1}{2}\mu(f^n(T_n))^\alpha\right) \le \exp\left(-\frac{1}{2}E_\alpha^n(\mu(S_0))^\alpha\right).\end{aligned}$$

We use this to prove that the set $T_n \setminus (\bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1})$ has small density in T_n. For all $k \in \{1, ..., n\}$, there is a square $S_k \in \mathcal{S}$ such that $f^k(T_n) \subset S_k$. In particular, f^{n+1} is injective in T_n. Thus

$$\begin{aligned}&\operatorname{meas}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}\right) \\ &\le \frac{1}{\mu(T_n, (f^{n+1})')^2} \cdot \operatorname{meas}\left(f^{n+1}(T_n) \setminus \bigcup_{S' \in \operatorname{pack}(f^{n+1}(T_n))} S'\right)\end{aligned}$$

and

$$\operatorname{meas}(T_n) \ge \frac{1}{\mathcal{M}(T_n, (f^{n+1})')^2} \cdot \operatorname{meas}(f^{n+1}(T_n)).$$

Hence

$$\begin{aligned}&\operatorname{dens}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1},\, T_n\right) \\ &\le \frac{\mathcal{M}(T_n, (f^{n+1})')^2}{\mu(T_n, (f^{n+1})')^2} \cdot \operatorname{dens}\left(f^{n+1}(T_n) \setminus \bigcup_{S' \in \operatorname{pack}(f^{n+1}(T_n))} S',\, f^{n+1}(T_n)\right) \\ &\le \left(\frac{\mathcal{M}(T_n, (f^{n+1})')}{\mu(T_n, (f^{n+1})')}\right)^2 \cdot \exp\left(-\frac{1}{2}E_\alpha^n(\mu(S_0))^\alpha\right). \qquad (5.4.6)\end{aligned}$$

To estimate $\mathcal{M}(T_n, (f^{n+1})')/\mu(T_n, (f^{n+1})')$, let $k \in \{0, ..., n\}$ and $w_0 \in f^k(T_n)$. Then

$$f^k(T_n) \subset S_k \subset \overline{\mathcal{D}(w_0, \sigma|w_0|^{-(d-1)})}.$$

By Lemma 5.3.1, f is injective in $\mathcal{D}(w_0, 2\sigma|w_0|^{-(d-1)})$. The Koebe distortion theorem yields that for all $w \in f^k(T_n)$, we have

$$\frac{|f(w) - f(w_0)|}{|w - w_0|} \geq \frac{4}{9}|f'(w_0)|.$$

By Lemma 5.2.5, $|f'(w_0)| \geq 5$. Thus

$$\operatorname{diam}(f^{k+1}(T_n)) > 2 \operatorname{diam}(f^k(T_n)).$$

Induction yields

$$\begin{aligned}\operatorname{diam}(f^k(T_n)) &< \frac{1}{2^{n-k}} \operatorname{diam}(f^n(T_n)) \leq \frac{1}{2^{n-k}} \cdot \frac{\sigma}{\mathcal{M}(f^n(T_n))^{d-1}} \\ &\leq \frac{1}{2^{n-k}} \cdot \frac{\sigma}{\mathcal{M}(f^k(T_n))^{d-1}}.\end{aligned}$$

In particular, for $z_k \in f^k(T_n)$, we have

$$f^k(T_n) \subset \overline{\mathcal{D}\left(z_k, \frac{2^{k-n}\sigma}{\mathcal{M}(f^k(T_n))^{d-1}}\right)} \subset \overline{\mathcal{D}(z_k, 2^{k-n}\sigma|z_k|^{-(d-1)})}.$$

Since f is injective in $\mathcal{D}(z_k, 2\sigma|z_k|^{-(d-1)})$, the Koebe distortion theorem yields

$$\frac{\mathcal{M}(f^k(T_n), f')}{\mu(f^k(T_n), f')} \leq \left(\frac{1 + 2^{k-n-1}}{1 - 2^{k-n-1}}\right)^4.$$

Because $(f^{n+1})'(z) = \prod_{k=0}^n f'(f^k(z))$, we deduce

$$\begin{aligned}\frac{\mathcal{M}(T_n, (f^{n+1})')}{\mu(T_n, (f^{n+1})')} &\leq \prod_{k=0}^{n} \left(\frac{1 + 2^{k-n-1}}{1 - 2^{k-n-1}}\right)^4 = \left(\prod_{j=1}^{n+1} \frac{1 + 2^{-j}}{1 - 2^{-j}}\right)^4 \\ &\leq \left(\prod_{j=1}^{\infty} \frac{1 + 2^{-j}}{1 - 2^{-j}}\right)^4 =: C_2,\end{aligned}$$

where $C_2 \in (0, \infty)$. Together with (5.4.6), this implies

$$\operatorname{dens}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}, T_n\right) \leq C_2^2 \exp\left(-\frac{1}{2} E_\alpha^n(\mu(S_0))^\alpha\right).$$

Thus

$$\begin{aligned}\operatorname{meas}(S_0 \setminus T) &= \sum_{n=0}^{\infty} \sum_{T_n \in \mathcal{K}_n} \operatorname{meas}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}\right) \\ &= \sum_{n=0}^{\infty} \sum_{T_n \in \mathcal{K}_n} \operatorname{dens}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}, T_n\right) \cdot \operatorname{meas}(T_n) \\ &\leq \sum_{n=0}^{\infty} C_2^2 \exp\left(-\frac{1}{2} E_\alpha^n(\mu(S_0))^\alpha\right) \sum_{T_n \in \mathcal{K}_n} \operatorname{meas}(T_n) \\ &\leq C_2^2 \sum_{n=0}^{\infty} \exp\left(-\frac{1}{2} E_\alpha^n(\mu(S_0))^\alpha\right) \cdot \operatorname{meas}(S_0).\end{aligned}$$

For all large x, we have $\exp(\alpha x^\alpha) \geq x^\alpha + 2\log 2$, and hence

$$\exp\left(-\frac{1}{2}E_\alpha(x)^\alpha\right) = \exp\left(-\frac{1}{2}\exp(\alpha x^\alpha)\right) \leq \frac{1}{2}\exp\left(-\frac{1}{2}x^\alpha\right).$$

Induction yields

$$\sum_{n=0}^{\infty}\exp\left(-\frac{1}{2}E_\alpha^n(\mu(S_0))^\alpha\right) \leq \sum_{n=0}^{\infty}\frac{1}{2^n}\exp\left(-\frac{1}{2}\mu(S_0)^\alpha\right) = 2\exp\left(-\frac{1}{2}\mu(S_0)^\alpha\right).$$

Thus

$$\operatorname{meas}(S_0 \setminus \mathcal{J}_\mathcal{A}) \leq \operatorname{meas}(S_0 \setminus T) \leq 2C_2^2\exp\left(-\frac{1}{2}\mu(S_0)^\alpha\right) \cdot \operatorname{meas}(S_0).$$

To conclude that the Lebesgue measure of $\mathbb{C} \setminus \mathcal{J}_\mathcal{A}$ is finite, fix $R > 0$ such that $B_0 \subset \mathcal{D}(0, R/2)$. For $r \geq R$, define

$$\operatorname{ann}(r) := \{z \in \mathbb{C} : r \leq |z| \leq 2r\}.$$

Then

$$\bigcup_{\substack{S_0\in\mathcal{S}\\ S_0\cap\operatorname{ann}(r)\neq\emptyset}} S_0 \subset \left\{z \in \mathbb{C} : \frac{r}{2} \leq |z| \leq 3r\right\} \subset \overline{\mathcal{D}(0, 3r)}.$$

We get

$$\begin{aligned}
\operatorname{meas}(\operatorname{ann}(r) \setminus (\mathcal{J}_\mathcal{A} \cup \mathcal{E}_2)) &\leq \sum_{\substack{S_0\in\mathcal{S}\\ S_0\cap\operatorname{ann}(r)\neq\emptyset}} 2C_2^2\exp\left(-\frac{1}{2}\mu(S_0)^\alpha\right)\operatorname{meas}(S_0)\\
&\leq 2C_2^2\exp\left(-\frac{1}{2}\left(\frac{r}{2}\right)^\alpha\right)\sum_{\substack{S_0\in\mathcal{S}\\ S_0\cap\operatorname{ann}(r)\neq\emptyset}}\operatorname{meas}(S_0) \leq 2C_2^2\exp\left(-\frac{r^\alpha}{2^{1+\alpha}}\right)\operatorname{meas}(\mathcal{D}(0,3r))\\
&= 18\pi C_2^2 r^2\exp\left(-\frac{r^\alpha}{2^{1+\alpha}}\right) \leq \exp\left(-\frac{r^\alpha}{2^{2+\alpha}}\right)
\end{aligned}$$

if r is sufficiently large. Applying this to the annuli $\operatorname{ann}(2^n R)$ for $n \geq 0$ yields

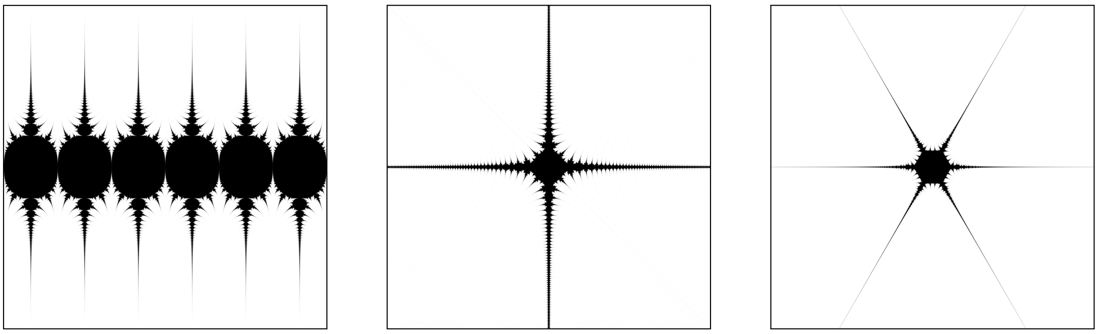
$$\begin{aligned}
\operatorname{meas}(\mathbb{C} \setminus \mathcal{J}_\mathcal{A}) &\leq \operatorname{meas}(\mathcal{D}(0,R)) + \operatorname{meas}(\mathcal{E}_2) + \sum_{n=0}^{\infty}\operatorname{meas}(\operatorname{ann}(2^n R) \setminus (\mathcal{J}_\mathcal{A} \cup \mathcal{E}_2))\\
&\leq \operatorname{meas}(\mathcal{D}(0,R)) + \operatorname{meas}(\mathcal{E}_2) + \sum_{n=0}^{\infty}\exp\left(-\frac{R^\alpha}{2^{2+\alpha}}2^{n\alpha}\right) < \infty. \qquad \square
\end{aligned}$$

5.5 Counterexamples

In this section we discuss examples that illustrate the necessity of the assumptions of Theorem E.

By Theorem E, the function $f(z) = \sin(z^d)$ with $d \geq 3$ satisfies $\operatorname{meas}(\mathbb{C} \setminus (\mathcal{J}(f) \cap \mathcal{A}(f))) < \infty$. As already mentioned in the introduction, this is not true for $d \in \{1, 2\}$. See Figure 5.3 for the Fatou sets of $\sin(z)$, $\sin(z^2)$ and $\sin(z^3)$.

Example 5.5.1. The Fatou set and the non-escaping set of $\sin(z)$ and $\sin(z^2)$ have infinite Lebesgue measure.



(a) $f(z) = \sin(z)$ (b) $f(z) = \sin(z^2)$ (c) $f(z) = \sin(z^3)$

Figure 5.3: The Fatou set of f, drawn in black, in the range $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 3\pi, |\operatorname{Im} z| \leq 3\pi\}$.

Proof. First let $f(z) = \sin(z)$. Because $f(0) = 0$ and $f'(0) = 1$, the function f has a parabolic domain where $f^n(z)$ tends to zero. In particular, the Fatou set of f is non-empty. Since $\mathcal{F}(f)$ is open and $\mathcal{F}(f)$ and $\mathcal{I}(f)$ are 2π-periodic, the Lebesgue measure of $\mathcal{F}(f)$ and $\mathbb{C} \setminus \mathcal{I}(f)$ is infinite.

Let us now consider the function $f(z) = \sin(z^2)$. Note that f has a superattracting fixed point at zero. Let $\varepsilon > 0$ such that $\mathcal{D}(0, \varepsilon)$ is contained in the attractive basin of zero. There exists $\delta \in (0, \pi/2)$ such that $\sin(\mathcal{D}(\pi k, \delta)) \subset \mathcal{D}(0, \varepsilon)$ for all $k \in \mathbb{Z}$. Let $D_k := \mathcal{D}(\pi k, \delta)$ and $p(z) = z^2$. Then $p^{-1}(D_k)$ is contained in the attractive basin of zero of the function f. We have

$$\operatorname{meas}(p^{-1}(D_k)) = 2 \int_{D_k} \left(\frac{1}{2\sqrt{|z|}} \right)^2 dx\, dy \geq \frac{1}{2(|k|\pi + \delta)} \operatorname{meas}(D_k) = \frac{\pi \delta^2}{2(|k|\pi + \delta)}.$$

Summing up over all k yields that the attractive basin of zero has infinite measure. □

Remark 5.5.2. Analogous arguments as those in the proof of Example 5.5.1 show that if $f(z) = a \exp(p(z)) + b \exp(-p(z))$, where $a, b \in \mathbb{C} \setminus \{0\}$ and p is a polynomial with $\deg(p) \in \{1, 2\}$, then either $\mathcal{J}(f) = \mathbb{C}$ or $\operatorname{meas}(\mathcal{F}(f)) = \infty$.

Let us now consider the function

$$h(z) = \frac{1}{2} \exp(z^3 + iz) - \frac{1}{2} \exp(-z^3 + iz) = \exp(iz) \sinh(z^3)$$

given in Example 5.1.1. Note that h satisfies all assumptions of Theorem E except for condition (5.1.1). Moreover, the function h has a superattracting fixed point at zero. Recall that we want to prove that the attractive basin of zero has infinite Lebesgue measure, so that in particular the Lebesgue measure of $\mathbb{C} \setminus (\mathcal{J}(h) \cap \mathcal{A}(h))$ is infinite. Figure 5.4 shows the attractive basin of zero of h.

Proof of Example 5.1.1. Fix $\varepsilon > 0$ such that $\mathcal{D}(0, \varepsilon)$ is contained in the attractive basin of zero. For large $r_0 > 1$, let

$$B := \left\{ re^{i\theta} : r \geq r_0, \left| \theta - \frac{\pi}{2} \right| \leq \frac{1}{r^2 \log r} \right\}.$$

We have

$$\operatorname{meas}(B) = \int_{r_0}^{\infty} \int_{\pi/2 - 1/(r^2 \log r)}^{\pi/2 + 1/(r^2 \log r)} r\, d\theta dr = \int_{r_0}^{\infty} \frac{2}{r \log r} dr = 2 \int_{\log(r_0)}^{\infty} \frac{1}{u}\, du = \infty.$$

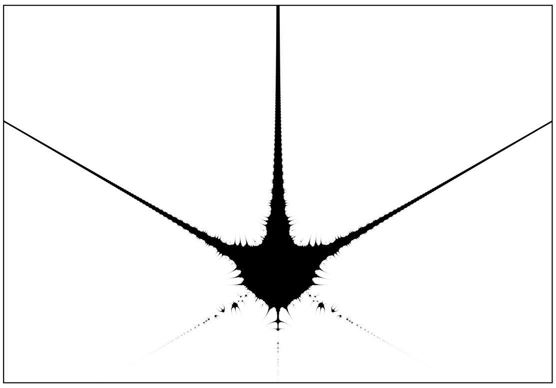


Figure 5.4: The attractive basin of zero of the function $h(z) = \exp(iz)\sinh(z^3)$. The displayed range is $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 8, -3 \leq \operatorname{Im} z \leq 8\}$.

We now show that if r_0 is sufficiently large, then $h(B) \subset \mathcal{D}(0,\varepsilon)$, and hence B is contained in the attractive basin of zero. Let $z = re^{i\theta} \in B$. Then

$$\begin{aligned}|h(z)| &\leq |\exp(iz)| \cdot \frac{1}{2}(|\exp(z^3)| + |\exp(-z^3)|)\\ &= \exp(-r\sin(\theta)) \cdot \frac{1}{2}(\exp(r^3\cos(3\theta)) + \exp(-r^3\cos(3\theta)))\\ &\leq \exp(-r\sin(\theta)) \cdot \exp(r^3|\cos(3\theta)|).\end{aligned}$$

We have

$$\sin(w) = 1 + O\left(\left(w - \frac{\pi}{2}\right)^2\right) \qquad \text{as } w \to \frac{\pi}{2}$$

and

$$\cos(w) = \left(w - \frac{3\pi}{2}\right) + O\left(\left(w - \frac{3\pi}{2}\right)^3\right) \qquad \text{as } w \to \frac{3\pi}{2}.$$

Since $|\theta - \pi/2| \leq 1/(r^2\log r)$, this implies

$$-\sin(\theta) \leq -\frac{1}{2}$$

and

$$|\cos(3\theta)| \leq 2 \cdot \frac{3}{r^2\log r}$$

if r is sufficiently large. Thus

$$|h(z)| \leq \exp\left(-\frac{1}{2}r\right) \cdot \exp\left(6 \cdot \frac{r}{\log r}\right) \leq \exp\left(-\frac{1}{4}r\right) < \varepsilon$$

if r is sufficiently large. So $h(B) \subset \mathcal{D}(0,\varepsilon) \subset \mathcal{F}(h) \setminus \mathcal{I}(h)$ if r_0 is sufficiently large. $\square$

Nomenclature

General notation

$\mathcal{A}(f)$	Fast escaping set of f, page 12
$\mathcal{A}(z_0)$	Attractive basin of the periodic point z_0, page 13
$\mathcal{A}^*(z_0)$	Immediate attractive basin of the periodic point z_0, page 13
$\mathcal{D}(f)$	Domain of definition of f, page 10
$\mathcal{D}(z_0, r)$	Open disk centred at z_0 with radius r, page 7
$\mathcal{F}(f)$	Fatou set of f, page 11
$\mathcal{I}(f)$	Escaping set of f, page 11
$\mathcal{J}(f)$	Julia set of f, page 11
$\mathcal{O}^{\pm}(A)$	Forward and backward orbit of A, respectively, page 12
$\mathcal{P}(f)$	Postsingular set of f, page 14
$\mathrm{dens}(A, B)$	Density of A in B, page 7
$\mathrm{diam}(A)$	Euclidean diameter of A, page 7
$\mathrm{dist}(A, B)$	Euclidean distance between A and B, page 7
$\hat{\mathbb{C}}$	Riemann sphere, page 7
$\mathrm{length}(\gamma)$	Euclidean length of γ, page 9
$\mathrm{meas}(A)$	Lebesgue measure of A, page 7
$\mathrm{sing}(f^{-1})$	Set of singular values of f in $\mathcal{D}(f)$, page 14
E_α	$E_\alpha(x) = \exp(x^\alpha)$, page 15
$M(r, f)$	Maximum modulus of f, page 12

Notation used in Chapter 4

$\mathcal{F}_j$	$\mathcal{F}(f) \cap \mathcal{S}_j$, page 38
$\mathcal{G}$	$\mathbb{C} \setminus (\overline{\mathcal{D}(0, R)} \cup [0, \infty))$, page 27
$\mathcal{H}(\mu, \alpha, \nu)$	$\{w \in \mathbb{C} : \operatorname{Re} w \geq \mu \log \lvert w\rvert - \log \alpha,\ \lvert \operatorname{Im} w\rvert \geq \nu\}$, page 33
$\mathcal{S}_j$	Component of $q^{-1}(\mathcal{G})$ and hence a domain where q is injective, page 27

$\Gamma(\mu,\alpha)$	$\{w \in \mathbb{C} : \operatorname{Re} w = \mu \log\lvert w\rvert - \log\alpha,\ \lvert\operatorname{Im} w\rvert \geq 2\lvert\mu\rvert\}$, the left boundary of $\mathcal{H}(\mu,\alpha,2\lvert\mu\rvert)$, page 33
λ	$(d-1-m)/d$, page 28
φ_j	Branch of q^{-1} that maps $\mathcal{G}$ to $\mathcal{S}_j$, page 28
c^*	$\max_{1\leq l\leq d}\lvert c_l\rvert$, page 50
c_j	Finite asymptotic value of g with asymptotic path in $\mathcal{S}_j$, page 28
d	Degree of q, page 27
h_j	Function $q\circ f\circ\varphi_j$ conjugate to f, page 39
m	Degree of p, page 27

Notation used in Chapter 5

$\mathcal{E}_l$	$\bigcup_{j\neq k} P_{j,k}^{-1}(\mathcal{U}_l)$, page 70
$\mathcal{U}_l$	$\{w \in \mathbb{C} : \lvert\operatorname{Re} w\rvert < l\lvert w\rvert^{\nu/d}\}$, page 70
ν	$d-(5/2)$, page 70
$P_{j,k}$	$P_{j,k}(z) = (b_j - b_k)z^d + (P_j(z) - P_k(z))$, page 70

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit – abgesehen von der Beratung durch meinen Betreuer – nach Inhalt und Form eigenständig und unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft verfasst habe und nur die angegebenen Hilfsmittel verwendet habe. Weiter erkläre ich, dass mir kein akademischer Grad entzogen wurde. Diese Arbeit hat weder ganz noch zum Teil an anderer Stelle im Rahmen eines Prüfungsverfahrens vorgelegen. Die Hauptergebnisse dieser Arbeit sind in folgenden Artikeln enthalten:

- M. Wolff, *Exponential polynomials with Fatou and non-escaping sets of finite Lebesgue measure*. Preprint arXiv:1908.03037. Online erschienen in *Ergodic Theory and Dynamical Systems*, doi:10.1017/etds.2020.120.
- M. Wolff, *A class of Newton maps with Julia sets of Lebesgue measure zero*. Preprint arXiv:2101.08045. Zur Veröffentlichung eingereicht.

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